Coalescence and instability of copropagating nonlinear waves

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An arbitrary number of light waves that collinearly propagate in a Kerr cubic medium is investigated in the framework of n ($n\geqslant 2$) coupled nonlinear Schrödinger equations. Depending on their initial separation distance and their power, the waves are shown to either disperse, collapse individually, or still attract each other to form a central lobe that may blow up at a finite time. General results, including the fundamental relations that govern the wave centroids and their mean square radii, are established for two and more light pulses. Their approximate evolution is described by means of a variational approach applied to two Gaussian beams and theoretical arguments detailing the attractor associated with the self-attraction of beams are also given. Furthermore, an instability criterion for coupled bound states is derived using perturbation theory. It is shown that coupled stationary-wave solutions are unstable when the space dimension number is higher than 2, while their corresponding ground states are stable at lower dimension. Finally, the competition between the modulational instability of coupled waves and their natural tendency to amalgamate into one self-focusing structure is discussed. [S1063-651X(98)13910-7]

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I. INTRODUCTION

For more than two decades, the question of optimizing the propagation of light beams in nonlinear bulk media has raised an increasing interest connected with the recent development of high-power laser sources [1,2]. Among the various nonlinear effects induced by the response of a nonlinear medium, the self-trapping and self-focusing of intense beams have been widely investigated since the 1960s. Self-trapped beams evolving in three spatial dimensions are well known to be unstable within an ideal Kerr material and to break up into a train of three-dimensional (3D) solitary waves [3,4]. In the absence of saturation, the resulting filaments self-focus until they collapse at a finite propagation distance, when their individual power exceeds a critical value such that the nonlinear effects continuously dominate over the natural dispersion of the wave [5]. This critical power is usually computed in the framework of the paraxial cubic model [3], based on the 2D nonlinear Schrödinger (NLS) equation with two transverse dimensions and one longitudinal dimension for the propagation axis. The dimension number D here corresponds to the one conventionally associated with the transverse plane ($D=2$), where wave diffraction takes place. Nevertheless, it can include a third dimension ($D=3$) to account for the variations of the wave field with respect to a retarded time variable, when the group velocity dispersion (GVD) of the wave is retained. In a physical medium, once self-focusing is initiated, the local wave intensity starts to increase and collapse is arrested by saturation of the Kerr response, which allows for the formation of stationary solitary waves. In the presence of anomalous GVD, these solitary waves form robust ''optical bullets'' [6], which consist of self-focused beamlets relaxing asymptotically to stable steady states.

Recently, the topic of interacting light waves opened a wide area of investigations: For instance, attraction and fusion of in-phase optical bright spatial solitons were displayed numerically [7] and experimentally [8]. From the theoretical side, analogous behaviors based on nonlinearities promoting wave collapse were revealed by numerical integrations of the cubic Schrödinger equation [9] and analytical arguments clearing them up were displayed in [10], when several beamlets compose the solution to one NLS equation. In particular, according to their initial powers and separation distance, two in-phase Gaussian beamlets symmetrically located at different centroids were shown to spread out, self-focus without mutual correlation, or fuse into a central self-focusing lobe. In this configuration, the coalescence mechanism is then initiated when the two beamlets are mutually separated by a distance less than a critical value depending on their individual power and when the latter is lying between $N_{\rm crit}/4$ and $N_{\rm crit}$. Here $N_{\rm crit}=4\pi$ denotes the threshold power for self-focusing of one Gaussian beam with a nonlinearity coefficient equal to unity. This critical threshold $N_{\rm crit}=4\pi$ is slightly larger than the well-known self-focusing power $N_c=11.68$ that a solution to the 2D NLS equation must initially exceed to collapse. Furthermore, by virtue of the conservation of power and the Hamiltonian, two initially separated beamlets with a π-phase difference were shown to never amalgamate into a single structure. These different behaviors were supported by numerical simulations.

Among the numerous physical processes dealing with the self-focusing of coupled light waves, we can here mention the self-excitation of waves with different polarizations in a nonlinear medium. From the pioneering works by Berkhoer and Zakharov [11] and Manakov [12], it is well established that, unlike circularly polarized beams, a wave having an arbitrarily varying polarization decomposes into two waves with opposite circular polarizations in an isotropic nongyrotropic medium. Under the assumption that the material's refractive index changes linearly with the optical intensity (the so-called Kerr effect), the slowly varying complex envelopes of the resulting waves obey in that case two distinct, nonlinearly coupled nonlinear Schrödinger equations in the form

$$i\partial_z E_1+\vec{\nabla}_\perp^2 E_1+(\Lambda_{11}|E_1|^2+\Lambda_{21}|E_2|^2)E_1=0, \qquad (1)$$

$$i\partial_z E_2+\vec{\nabla}_\perp^2 E_2+(\Lambda_{12}|E_1|^2+\Lambda_{22}|E_2|^2)E_2=0, \qquad (2)$$

which can easily be derived by assuming the space and time envelope approximations. Due to its vectorial nature, a wave with electric field $\vec{E}=E_1\vec{c}_R+E_2\vec{c}_L$ ($\vec{c}_R$ and $\vec{c}_L$ denote the complex unit vectors corresponding to orthogonal polarizations) is then stratified into beams with a constant specific polarization of radiation in such a way that the cubic medium promotes the formation of waveguide channels. In the above equations, the first term accounts for the propagation of the light electric field envelopes $\vec{E}_1(\vec{r}_\perp,z)$ and $\vec{E}_2(\vec{r}_\perp,z)$ along the z axis, expressed in the frame moving with the group velocity of the beam. The second term describes the transverse diffraction of the wave components and the third one represents the cubic nonlinearity induced by the medium response with nonlinearity coefficients $\Lambda_{\alpha\beta}$ $(\alpha,\beta=1,2)$. Here the transverse Laplacian reads $\vec{\nabla}_\perp^2\equiv\partial_x^2+\partial_y^2$ for a diffraction plane spanned by the radius $\vec{r}_\perp=(x,y)$. For technical convenience, this diffraction plane may formally be extended to D transverse dimensions, including a temporal dimension to account for anomalous GVD.

Equations (1) and (2) generally apply to incoherent waves with the same central frequency, for which the intensities are simply added, so that the nonlinearity coefficients may be identical. They can be extended to an infinite set of coupled nonlinear Schrödinger equations and serve, for instance, as a theoretical model describing the self-trapping and self-focusing of incoherent light beams with low-intensity profiles in nonsaturating biased photorefractives [13]. In the one-dimensional case $(D=1)$, the formation of solitons was earlier discovered by Manakov [12] on the basis of the same equations. The properties of the soliton solutions for this model followed from a direct application of the inverse scattering transform (IST) techniques and the potential intersections of several wave forms, involving more than two soliton solutions, were also investigated. Let us here recall that the integration of system (1) through the IST method can be performed under specific conditions involving a low dimensionality $(D<2)$ and severe constraints on the nonlinearity coefficients. From a physical viewpoint, these constraints amount to imposing that the ratio between the components of self- and cross-phase modulations intervening in the coefficients $\Lambda_{\alpha\beta}$'s must be equal to the unity, while the self-phase modulation components need to be identical for the two polarizations. In addition, four-wave mixing (FWM) contributions that parametrically mix wave components must be zero. This condition can easily be fulfilled if the two orthogonally polarized beams are incoherent so as to average the FWM terms to zero, which can be made possible in, e.g., an AlGaAs planar waveguide [14]. In connection with this topic, the formation and stability of solitons in birefringent materials, such as single-mode optical fibers, were numerically investigated since linear birefringence leads to splitting an input pulse into two polarization modes. In the presence of the Kerr nonlinearity, it was observed that the fractional pulses in each of the two polarizations may trap each other and move together provided the soliton amplitudes exceed a threshold value. Without FWM, the two partial pulses lock together and travel as one unit when their amplitude is initially equal and above some threshold whose size increases with birefringence [15,16]. In the presence of FWM terms, families of soliton solutions can be constructed for birefringent optical fibers by taking radiation processes into account, which was shown to qualitatively influence the soliton solutions [17]. Coupled wave equations describing comparable phenomena were also derived for monomode step-index optical fibers in [18].

In the previous investigations, finding exact soliton solutions and identifying the role of radiation for given initial waves are possible because the propagation equations involve a transverse diffraction plane with low-dimension number $D=1$. In this situation only, the techniques of inverse scattering transform can be applied to nonlinear integrable systems, such that not only the emergence of solitons but also the radiative contributions can be quantified. As an alternative method, the so-called variational approach, or average Lagrangian method, in which the solution is approximated by a trial solution (ansatz) with a sech or Gaussian test function, may supply relatively good results for describing the localized core of the solution [19]. However, it never describes the amount of power dissipated through radiation, which can play a significant role during the soliton evolution. Recently, radiative corrections to solutions of the standard variational approach were proposed by Kath and Smyth [20] in order to remove the discrepancies introduced by this approximation method in the treatment of the 1D NLS equation for a single wave. They indeed showed that by supplementing a z-dependent chirped-sech core with radiative losses, the soliton core could attain a mean size with steady amplitude, whereas these quantities, when they are basically computed from the variational method, periodically oscillate around the mean soliton size. Without such radiative corrections, the standard variational approach was also applied to the 1D version of the coupled NLS equations (1) and (2) with an ansatz containing a finite number of dynamical parameters, which included the amplitude, the size, the frequency chirp, the velocity, and an arbitrary phase in sech-type soliton solutions [21,22]. Through this method, it was speculated that bound states can emerge from identical, symmetrically located pulses, when their initial velocity is not too large, and form a unique bound state for which the soliton widths and positions oscillate around mean values. Also, in spite of discrepancies connected with radiation emission, both numerics and average Lagrangian formalism displayed evidence that solitons, with initial velocities v_0 sufficiently large $(v_0>v_e)$ to overcome a mutual attraction, could escape from each other and become well separated asymptotically. The critical velocity v_e permitting this escape process was shown to increase with the soliton amplitude.

On the other hand, comparable investigations were performed by McKinstrie and co-workers [23,24] in the context of two light waves propagating in a plasma beat-wave accelerator, with emphasis on the physical configuration $D=2$. This specific situation is in analogy with the former equations since the scalar envelopes of such light waves evolve according to the coupled NLS equations

$$i\partial_t A_\alpha+\vec{\nabla}^2 A_\alpha+Q_\alpha A_\alpha=0. \qquad (3)$$

Here the time variable t plays the role of the propagation distance z, compared with Eqs. (1) and (2), and Q_α

$\equiv \partial Q/\partial|A_\alpha|^2$ denotes the derivative of a potential function Q with respect to the αth wave intensity. For a weakly relativistic electron quiver velocity, two light waves are expected to couple through the nonlinear potential

$$Q=\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Lambda_{\alpha\beta}|A_\alpha|^2|A_\beta|^2, \tag{4}$$

where the values of the nonlinearity coefficients depend on the polarization state of the waves. When assuming a circular polarization and initially scalar wave fields, the original system (3) can readily be extended to the nonlinear coupling of n $(n\geqslant 2)$ scalar wave envelopes. For two waves only, the nonlinearity coefficients are symmetric with the values $\Lambda_{11}=\Lambda_{22}=1/8$ for the self-interaction components and with $\Lambda_{12}=\Lambda_{21}=1/4$ for the crossed components. Studying the modulational instability of plane waves in the form $A_\alpha(\vec{x},t)=A_\alpha^0\exp[i(\Sigma_\beta\Lambda_{\alpha\beta}|A_\beta|^2)t]$ by periodic perturbations, McKinstrie and Bingham [23] established the growth rate of the instability of n coupled waves

$$\gamma_\alpha^2(\vec{k})=k_j^2(2\Delta_\alpha-k_j^2), \tag{5}$$

where k_j $(j=1,...,D)$ denotes the components of the wave vector carried by the perturbation and the Δ_α's are the eigenvalues of the stability matrix

$$\begin{pmatrix} \Lambda_{11}|A_1|^2 & \Lambda_{12}|A_1A_2| & \cdots & \Lambda_{1n}|A_1A_n| \\ \Lambda_{21}|A_2A_1| & \Lambda_{22}|A_2|^2 & \cdots & \Lambda_{2n}|A_2A_n| \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_{n1}|A_nA_1| & \Lambda_{n2}|A_nA_2| & \cdots & \Lambda_{nn}|A_n|^2 \end{pmatrix}. \tag{6}$$

These driving terms Δ_α depend on a symmetric combination of the initial amplitudes of both waves and for each spatial eigenfunction of the perturbative modes there is a pair of temporal growth rates $\gamma_\alpha^2(\vec{k})$ associated with each eigenvalue of the stability matrix entering the linearized problem of Eq. (3). In particular, for two waves, Δ_α reads [23]

$$2\Delta_\pm=(\Lambda_{11}|A_1|^2+\Lambda_{22}|A_2|^2) \pm\sqrt{(\Lambda_{11}|A_1|^2-\Lambda_{22}|A_2|^2)^2+4\Lambda_{12}\Lambda_{21}|A_1A_2|^2}. \tag{7}$$

Thus, whenever the system of coupled waves is unstable (which is ensured for $\Lambda_{12}^2>|\Lambda_{11}\Lambda_{22}|$ with $\Lambda_{12}=\Lambda_{21}$), the maximal growth rate of the instability is given by $\gamma_{\max}=\Delta_+$ and corresponds to the optimal wave number $k_{\max}=\sqrt{\Delta_+}$ for the most unstable perturbative mode. Note that from Eq. (7) the nonlinear coupling of two waves contributes to enhance their instability since the coupling term in Eq. (7) participates in a positive amount, making the maximal growth rate larger than the growth rate of either wave alone.

Furthermore, nonlinearly coupled waves were studied from the viewpoint of their mutual interactions and self-focusing dynamics [24]. Regarding the occurrence of collapse, general results based on the main invariants of the system (3) and on virial-type arguments allowed for bringing to light an entrainment mechanism between two Gaussian waves being located at different centroids. Conditions for self-focusing of two superimposed waves were detailed in Ref. [24] (see also the more recent investigation [25]), while for initially separated waves, a critical distance between two identical waves was identified as the minimal distance beyond which they may behave independently of one another. Below this critical distance, when the power of each wave is larger than the third of the critical power for self-focusing for one Gaussian beam and smaller than this critical power threshold itself, two identical waves can attract each other. This entrainment mechanism is comparable to that of two beamlets composing one self-focusing beam governed by a single NLS equation [10]. Although originally discovered in Ref. [24], wave amalgamation, consisting in the coalescence of two waves fusing into a self-focusing central lobe, was revealed for Gaussian beams with a small separation distance between their centroids. Therefore, this interaction pattern still deserves to be investigated when coupled waves are initially well separated from one another by at least one wave diameter.

The aim of this paper is to generalize the previous results to the coupling of n $(n\geqslant 2)$ NLS wave packets in a cubic medium and to determine their regimes of mutual coalescence. We first recall in Sec. II the main properties of collapsing solutions for NLS equations with multiple wave components and we derive some criteria of the blow-up and coalescence of wave envelopes in terms of a critical separation distance depending on their individual powers. This yields general results for two or more light pulses, inferred from relations governing the centroids and the mean square radius of the waves. In Sec. III the approximate evolution of two light waves is described by means of a variational approach applied to Gaussian beams. The shape of the attractor attached to the self-attraction of beams modeled with Gaussians and sech functions is also discussed. In Sec. IV an instability criterion, derived from the so-called Vakhitov-Kolokolov criterion of stability for solitonlike bound states, is constructed by using straightforward perturbative techniques around the soliton states. This result emphasizes that for cubic nonlinearities bound states of coupled NLS equations are unstable whenever the space dimension number is greater than or equal to the critical value 2, which is consistent with the property following which solutions to this system can blow up in finite time when $D\geqslant 2$. Finally, in Sec. V the competition between the modulational instability of coupled waves, which tends to break them transversally during the early stage of their quasilinear evolution, and their natural tendency to form one self-focusing structure is discussed in terms of typical time scales along which the production of small-sized cells is favored. To conclude this work, the dynamics of counter- and copropagating beams are also briefly compared.

II. GENERAL RESULTS

To start with, we consider light waves that propagate with a constant polarization in a bulk cubic medium following the model equations (3). For a suitable optical material, the saturation effects in the form, e.g., $Q_\alpha=\Lambda|A_\alpha|^2/(1+a|A_\alpha|^2)$ for one wave [26,27], where a is the saturation coefficient, are expected to limit the singular growth of the wave and to form steady bullets in the medium. In the present investigation, however, we will ignore these saturation effects by formally taking the limit $a\rightarrow 0$ and permit the waves to collapse freely in self-focusing regimes. Although ''unphysical,'' a collapse-type singularity emerging in the medium will enable us to stress more the localization mechanism of the waves during their self-focusing stage. Thus the analysis presented below rather concerns waves growing in this stage before saturation becomes efficient.

A. Case of a single NLS wave

We first recall some basic properties attached to the nonlinear Schrödinger equation for one wave whose electric field envelope $u(\vec{x},t)$ is henceforth assumed to be scalar and governed by

$$i\partial_t u+\vec{\nabla}^2 u+\Lambda|u|^2u=0. \tag{8}$$

The positive constant Λ represents the nonlinearity coefficient and the time variable t has been chosen as a general evolution variable: It can denote a true time variable t in the scope of plasma physics [24], or a propagation distance z as well, when dealing with paraxial self-focusing of light [1,2]. In this case, the Laplacian operator $\vec{\nabla}^2\equiv\vec{\nabla}_\perp^2$ corresponds to the transverse diffraction of the waves taking place in a plane of vector $\vec{x}\equiv\vec{r}_\perp$. For technical convenience, we will sometimes employ the standard notations for denoting the L^p norms

$$\|f\|_p\equiv\left(\int|f|^p d\vec{x}\right)^{1/p}\Rightarrow\|f\|_p^p\equiv\int|f|^p d\vec{x}, \tag{9}$$

assuming any function f being L^p integrable. Also, in connection with the Cauchy problem associated with Eq. (8), we shall henceforth suppose that the wave function $u(\vec{x},t)$ evolves from initial data $u(\vec{x},0)\equiv u_0(\vec{x})$ belonging to the Sobolev space H^1 with norm $\|u\|_{H^1}=(\|u\|_2^2+\|\vec{\nabla}u\|_2^2)^{1/2}$, which invites us to search for solutions u localized in the transverse plane and decaying to zero at infinity.

Two main integrals of motion are related to the solutions of Eq. (8), namely, the dimensionless power

$$N\equiv\|u\|_2^2 \tag{10}$$

and the Hamiltonian

$$H\equiv\|\vec{\nabla}u\|_2^2-\frac{\Lambda}{2}\|u\|_4^4, \tag{11}$$

from which Eq. (8) can be derived through the Hamilton formulation $i\partial_t u=\delta H/\delta u^*$ (the asterisk means complex conjugate). In addition, Eq. (8) constitutes a Lagrangian system deriving from the Lagrangian integral

$$L=\frac{i}{2}\int(u^*\partial_t u-u\partial_t u^*)d\vec{x}-H, \tag{12}$$

from which a fundamental relation governing the mean square radius of solutions u can be established. This mean square radius, often called the virial integral, reads $I(t)\equiv\langle(\vec{x}-\langle\vec{x}\rangle)^2\rangle$, where the angular brackets refer to the mean value of any function $f(\vec{x})$ defined by the integral $\langle f(\vec{x})\rangle\equiv N^{-1}\int f(\vec{x})|u|^2d\vec{x}$. The relation for the evolution of this virial integral can be directly computed by first multiplying Eq. (8) by x^2u^* $(x=|\vec{x}|)$ and selecting the imaginary part of the space-integrated result and second by integrating over space the real part of Eq. (8) multiplied by $(\vec{x}\cdot\vec{\nabla}u^*)$. Taking the time derivative of the first relation finally yields [5]

$$\partial_t^2 I(t)=\frac{4}{N}\left\{2H_0+\frac{\Lambda}{2}(2-D)\|u\|_4^4\right\},\quad H_0\equiv H-\frac{N}{4}\langle\dot{\vec{x}}\rangle^2, \tag{13}$$

where $\langle\dot{\vec{x}}\rangle\equiv\partial_t\langle\vec{x}\rangle$ is the velocity of the center of mass of the localized wave packet. This velocity $\langle\dot{\vec{x}}\rangle$ is in turn governed by the relation $\partial_t\langle\vec{x}\rangle=\vec{P}/N$, where $\vec{P}\equiv 2\,\mathrm{Im}\int(u^*\vec{\nabla}u)d\vec{x}$ is the wave momentum. From the identity (13), one then infers that the mean square radius of a given localized wave tends to zero in a finite time, henceforth denoted by $t_c<+\infty$, under some specific conditions. Among those, the most well known ones are the condition of multidimensionality $D\geqslant 2$, and the requirement of having negative-energy states, $H_0<0$. Although other conditions involving nonzero initial divergences $[\partial_t I(0)<0]$ of the wave fields are available in the current literature [5,28], $H_0<0$, meaning that nonlinearities continuously dominate over wave dispersion, applies to any initial focusing shape of the waves and we shall retain it only for the sake of simplicity. So negative-energy wave forms centered on the origin with a zero velocity for their center of mass have a typical transversal scale tending to zero as $t\rightarrow t_c$ provided $D\geqslant 2$. In that case, the virial identity simply reduces to the inequality

$$\partial_t^2 I(t)\leqslant\frac{8H}{N},\quad I(t)\equiv\frac{\|\vec{x}u\|_2^2}{N}. \tag{14}$$

From the vanishing $I(t)\rightarrow 0$, one then deduces that the L^2 norm of the gradient of u must tend to infinity by virtue of the inequality [28]

$$\|u\|_2^2\leqslant\frac{2}{D}\|\vec{\nabla}u\|_2\|\vec{x}u\|_2, \tag{15}$$

which follows from a straightforward estimate of the norm N integrated by part. In this limit, the L^4 norm of u also diverges in turn because of the constancy of H and the wave blows up with a maximum of $|u|$ growing to infinity at the center [29]. These mathematical properties thereby reflect the singular nature of the collapse process originating from the vanishing of $I(t)$. Let us recall in this respect that a wave blow-up, characterized by the divergence $\|\vec{\nabla}u\|_2^2\rightarrow+\infty$, generally takes place before the complete vanishing of the virial integral $I(t)$, which yields only a maximum collapse time [28]. Indeed, wave blow-up occurs before the total vanishing of $I(t)$ because only a finite amount of power is captured from N in the collapse process. For instance, in the so-called critical case $D=2$, this finite amount of power is nothing but the critical power for self-focusing N_c/Λ with $N_c=11.68\simeq 11.7$, which corresponds to the smallest L^2 norm obtainable from the radially symmetric stationary solutions $u(\vec{x},t)=e^{it}\phi(|\vec{x}|)$ of Eq. (8). The solution corresponding to this norm is even, positive, and unique and N_c enters the following bound from beneath the Hamiltonian:

$$H \geqslant \|\vec{\nabla} u\|_2^2 \left(1 - \frac{\Lambda}{N_c} N\right), \tag{16}$$

which is established after using the Sobolev inequality $\|u\|_4^4 \leqslant C\|\vec{\nabla} u\|_2^2 \|u\|_2^2$ with the best constant $C_{\text{best}} = 2/N_c$ [30]. As a wave collapse is characterized by the divergence of the gradient norm of the envelope u, the requirement $N > N_c/\Lambda$ thus arises as a *necessary condition* for initiating the collapse.

B. Case of one NLS wave with several distinct components

Before discussing the interaction of copropagating nonlinear waves, we first investigate the case when one solution $u(\vec{x},t)$ of Eq. (8) is composed by n distinct ''beamlets'' (or ''cells'') initially well separated and each exhibiting a maximum located at the centroid $\langle \vec{x}_\alpha(t)\rangle$ $(\alpha = 1,...,n)$. Thus solutions to Eq. (8) can be sought under the form

$$u(\vec{x},t) = \sum_{\alpha=1}^{n} u_\alpha(\vec{x} - \langle \vec{x}_\alpha(t)\rangle, t), \tag{17}$$

where the discrete centers of mass $\langle \vec{x}_\alpha(t)\rangle$ are functions of time. For simplicity, we consider the simplest configuration dealing with two beamlets, i.e., $u(\vec{x},t) = u_1(\vec{x} - \langle \vec{x}_1(t)\rangle, t) + u_2(\vec{x} - \langle \vec{x}_2(t)\rangle, t)$. Only few analytical results can be inferred in such a case, apart from those deduced from the initial data themselves: By computing the total invariants N and H in terms of their ''free'' counterparts

$$N_\alpha \equiv \|u_\alpha\|_2^2, \quad H_\alpha \equiv \|\vec{\nabla} u_\alpha\|_2^2 - \frac{\Lambda}{2}\|u_\alpha\|_4^4 \quad (\alpha = 1,2), \tag{18}$$

supplemented by their respective interaction contributions, it is always possible from the values of N and H estimated with $u_0(\vec{x}) \equiv u(\vec{x},0)$ to speculate on the influence of their interaction contributions. We can indeed deduce whether the waves will either behave independently of each other or strongly interact mutually, depending on the size of their initial separation distance $\delta(0) \equiv |\langle \vec{x}_1(0)\rangle - \langle \vec{x}_2(0)\rangle|$ occurring in these interaction terms. For Gaussian beamlets, the interaction terms of H and N exponentially decrease with $\delta(0)$ in such a way that entrainment and amalgamation between two light cells into one central lobe can be promoted when $\delta(0)$ is smaller than a critical value ensuring that H remains of the order $H(\delta(0) = 0)$. In the following we give some necessary conditions on threshold powers, above which two separated beamlets can coalesce and collapse in finite time when $D = 2$. Plugging the decomposition $u = u_1 + u_2$ into the integral N, we share the different contributions of the latter as

$$N = N_1 + N_2 + 2\,\text{Re}\int (u_1 u_2^*) d\vec{x} \leqslant N_1 + N_2 + 2\int |u_1||u_2| d\vec{x} \tag{19}$$

and apply the Schwarz inequality together with the obvious estimate $2ab \leqslant a^2 + b^2$ to finally get

$$N \leqslant 2(N_1 + N_2). \tag{20}$$

By means of the bound (16) and the virial inequality (14), we therefore deduce that light cells with a zero initial velocity of their respective centers of mass can collapse under the necessary requirement $2(N_1 + N_2) > N_c/\Lambda$. Thus, for two identical beamlets $(N_1 = N_2)$, each of them must possess an individual power above the threshold

$$N_1 = N_2 > N_c/4\Lambda \tag{21}$$

to participate in the blow-up of the whole beam. In the opposite case with $N_1 = N_2 < N_c/4\Lambda$, the Hamiltonian integral H in Eq. (16) must be positive and the beam simply spreads out with two dispersing components. The bound (21) particularly applies to two superimposed cells with $\langle \vec{x}_1\rangle = \langle \vec{x}_2\rangle$ $[\delta(0) = 0]$. It then arises as the minimum power that each beamlet must engage in the best possible situation to favor the collapse, i.e., when both light cells participate in the collapse process by overlapping completely at one point in the transverse diffraction plane.

Let us now investigate the case of several (n) distinct beamlets that are well separated and identified by their respective maxima located at the centroids $\langle \vec{x}_\alpha(t)\rangle$ $(\alpha = 1,...,n)$. In this situation, the whole solution has the general form (17) and the total invariant N is expressed as

$$N = \int (u_1 + u_2 + \cdots + u_n)(u_1^* + u_2^* + \cdots + u_n^*) d\vec{x} \leqslant \int n(|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2) d\vec{x} = n \sum_{\alpha=1}^{n} \|u_\alpha\|_2^2, \tag{22}$$

after using Chebyshev's inequality. Thus, in view of Eq. (16), N must exceed N_c/Λ for initiating a wave blow-up and, consequently, we obtain the necessary condition for self-focusing $\sum_{\alpha=1}^{n} N_\alpha > N_c/n\Lambda$. So n identical beamlets must each possess an individual power above the critical threshold

$$N_1 = N_2 = \cdots = N_n > N_c/n^2\Lambda. \tag{23}$$

Note that the condition (23) is not sufficient to guarantee that the waves systematically undergo a collapse. Promoting a wave collapse also depends on the distribution of the initial wave forms and on the resultant sign of the total Hamiltonian intervening in the virial relation (13). However, from Eq. (22), it can be inferred that the constraint (23) on beamlet power, which yields a minimum power for reaching self-focusing regimes, can be lowered by increasing the number of beams: An infinity $(n \to +\infty)$ of equal beamlets with a very weak (almost zero) individual power could in principle promote wave collapse by superimposing efficiently their intensity distributions. This property can be transposed for n initially steady-state Gaussian beamlets composing the initial beam shape

$$u(\vec{x},0) = \sum_{\alpha=1}^{n} \left(\frac{N_\alpha}{\pi\rho_\alpha^2}\right)^{1/2} \exp\left[-\frac{[\vec{x} - \langle \vec{x}_\alpha(0)\rangle]^2}{2\rho_\alpha^2} + i\psi_\alpha\right], \tag{24}$$

all of them being superimposed on the origin of coordinates with $\langle \vec{x}_\alpha(0)\rangle = \vec{0}$. Here ρ_α is a measure of the mean radius of each Gaussian component and ψ_α an arbitrary constant phase factor. Regarding, for simplicity, light filaments with equal radius ($\rho_\alpha = \rho$) and zero phase ($\psi_\alpha = 0$), the total power contained within the entire beam reads $N = (\Sigma_{\alpha=1}^n \sqrt{N_\alpha})^2$, while the Hamiltonian H expands as $H = (N/\rho^2)(1 - \Lambda N/4\pi)$. The occurrence of collapse is then possible for $N > 4\pi/\Lambda$, ensuring thereby $H < 0$. Thus, for beamlets with equal power, we get the condition for each αth beamlet: $N_\alpha > 4\pi/\Lambda n^2$, which is compatible with Eq. (23). Collapse may then occur in this configuration for an infinite number of beamlets ($n \to +\infty$), each possessing a weak power ($N_\alpha \to 0$). For two Gaussian beamlets only, the necessary condition for promoting a wave collapse turns out to read $N_1 = N_2 \geqslant N_{\mathrm{crit}}/4\Lambda = \pi/\Lambda$, as discovered in [10].

This result can be generalized to beamlets having an arbitrary initial shape $\phi_\alpha(\vec{x},0)$ and entering the wave function $u(\vec{x},0) = \Sigma_{\alpha=1}^n \sqrt{N_\alpha}\phi_\alpha(\vec{x},0)$. The function ϕ_α can be chosen as, e.g., a sech function for dealing with a critical threshold power closer to 11.7 than 4π when $\Lambda = 1$ [28]. Considering initially superimposed beamlets, N is expressed as $N = (\Sigma_{\alpha=1}^n \sqrt{N_\alpha})^2 \|\phi_\alpha\|_2^2$ and H can be written as

$$H = \left(\sum_{\alpha=1}^n \sqrt{N_\alpha}\right)^2 \|\vec{\nabla}\phi_\alpha\|_2^2 \times \left(1 - \frac{\Lambda\left(\sum_{\alpha=1}^n \sqrt{N_\alpha}\right)^2}{2} \frac{\|\phi_\alpha\|_4^4}{\|\vec{\nabla}\phi_\alpha\|_2^2}\right).$$

Hence the constraint of, e.g., negative H for collapse implies that the power in each beamlet must together satisfy

$$\left(\sum_{\alpha=1}^n \sqrt{N_\alpha}\right)^2 > K \equiv \frac{2\|\vec{\nabla}\phi_\alpha\|_2^2}{\Lambda \|\phi_\alpha\|_4^4}, \tag{25}$$

which yields $N_\alpha > K/n^2$ for beamlets with equal power ($K = 4\pi/\Lambda$ in the Gaussian case).

C. Case of several copropagating waves

On the basis of the model equations (3) and (4), we now investigate the dynamical properties of $n \geqslant 2$ copropagating waves, with a particular attention to the possibilities of making them merge and self-focus in finite time. To this aim, we rewrite the set of equations (3), defined with the potential function $Q = \frac{1}{2}\Sigma_{\alpha,\beta}\Lambda_{\alpha\beta}|u_\alpha|^2|u_\beta|^2$, in the generic form

$$i\partial_t u_\alpha + \vec{\nabla}^2 u_\alpha + \sum_\beta \Lambda_{\alpha\beta}|u_\beta|^2 u_\alpha = 0, \tag{26}$$

where Σ with indices α and/or β refers to a summation over $1 \leqslant \alpha,\beta \leqslant n$ (for instance $\Sigma_{\alpha,\beta} \equiv \Sigma_{\alpha=1}^n \Sigma_{\beta=1}^n$). Unlike the preceding context for which only the total power $N = \|\Sigma_\alpha u_\alpha\|_2^2$ was preserved, the system (26) now conserves the individual powers $N_\alpha \equiv \|u_\alpha\|_2^2$ separately and the total Hamiltonian

$$H = \sum_\alpha \|\vec{\nabla} u_\alpha\|_2^2 - \sum_{\alpha,\beta} \frac{\Lambda_{\alpha\beta}}{2}\|u_\alpha u_\beta\|_2^2. \tag{27}$$

This directly follows from multiplying Eq. (26) by $\partial_t u_\alpha^*$ and integrating over space the real part of the result, after summing up over the entire set of indices (α,β). Conservation of H, proceeding from the conservation law $\partial_t H = 0$, requires one to assume symmetric nonlinearity coefficients satisfying $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$ for $\alpha \neq \beta$, which will be done in what follows. In that case, we can derive a dynamical relation for the total center of mass $\langle \vec{x}\rangle = N^{-1}\int \vec{x}\Sigma_\alpha |u_\alpha|^2 d\vec{x} N \equiv \Sigma_\alpha \|u_\alpha\|_2^2$. From straightforward algebra, one easily obtains

$$\partial_t \int \vec{x}|u_\alpha|^2 d\vec{x} = 2\,\mathrm{Im}\int (u_\alpha^* \vec{\nabla} u_\alpha)d\vec{x},$$
$$\partial_t\left[\mathrm{Im}\int (u_\alpha \vec{\nabla} u_\alpha^*)d\vec{x}\right] = -\int |u_\alpha|^2 \vec{\nabla}\sum_\beta \Lambda_{\alpha\beta}|u_\beta|^2 d\vec{x}, \tag{28}$$

from which the center of mass for the αth wave $\langle \vec{x}_\alpha(t)\rangle \equiv N_\alpha^{-1}\int \vec{x}|u_\alpha|^2 d\vec{x}$ is found to be governed by the relation

$$\partial_t^2 \langle \vec{x}_\alpha\rangle = \frac{2}{N_\alpha}\int |u_\alpha|^2 \vec{\nabla}\sum_{\beta\neq\alpha}\Lambda_{\alpha\beta}|u_\beta|^2 d\vec{x} = \frac{2}{N_\alpha}\int |u_\alpha|^2 \vec{\nabla} Q_\alpha d\vec{x}. \tag{29}$$

Therefore, the total center of mass obeys the identities

$$\partial_t \langle \vec{x}\rangle = \frac{\vec{P}}{N}, \quad \vec{P} = 2\,\mathrm{Im}\sum_\alpha \int (u_\alpha^* \vec{\nabla} u_\alpha)d\vec{x},$$
$$\partial_t^2 N\langle \vec{x}\rangle \equiv \partial_t^2 \sum_\alpha N_\alpha \langle \vec{x}_\alpha\rangle = \vec{0}, \tag{30}$$

which leads to the conservation of the total wave momentum $\vec{P}(t) = \vec{P}(0)$. If the initial data ensure $\vec{P} = \vec{0}$, one has $\langle \dot{\vec{x}}\rangle = 0$ (as for, e.g., Gaussians without space-varying phase) and $\langle \vec{x}(t)\rangle = \Sigma_\alpha N_\alpha \langle \vec{x}_\alpha(t)\rangle / N = \langle \vec{x}(0)\rangle$ is fixed at every time. Consequently, for two equal waves with a total center of mass located at the origin, the separation vector $\vec{\delta}(t) = \langle \vec{x}_1(t)\rangle - \langle \vec{x}_2(t)\rangle$ between their respective centroids will simply be given by $\vec{\delta}(t) = 2\langle \vec{x}_1(t)\rangle$. Furthermore, the wave identified by the subscript 1 has a center of mass evolving as

$$\partial_t^2 \langle \vec{x}_1(t)\rangle = \frac{2\Lambda_{12}}{N_1}\int |u_1|^2 \vec{\nabla}|u_2|^2 d\vec{x}, \tag{31}$$

where the right-hand side corresponds to the flux induced by wave 2 from the crossed contributions of the nonlinearity. By repeating the computational stages in Sec. II A, we can moreover establish a virial-type identity for the mean square radius

$$I(t)\equiv\frac{1}{N}\int(\vec{x}-\langle\vec{x}\rangle)^2\sum_\alpha|u_\alpha|^2d\vec{x} \tag{32}$$

consisting of a generalized form of Eq. (13), namely (see also [24]),

$$\partial_t^2I(t)=\frac{4}{N}\left\{2H_0+\int\left[2Q+D\left(Q-\sum_\alpha Q_\alpha|u_\alpha|^2\right)\right]d\vec{x}\right\}$$
$$=\frac{4}{N}\left\{2H_0+\left(\frac{2-D}{2}\right)\sum_{\alpha,\beta}\Lambda_{\alpha\beta}\|u_\alpha u_\beta\|_2^2\right\}, \tag{33}$$

with $H_0\equiv H-\frac{1}{4}\vec{P}^2/N$. Expression (33) reduces to the conventional form $\partial_t^2I(t)=8H/N$ in the 2D case for waves carrying a zero momentum. Henceforth regarding this critical case $D=2$, we use both Schwarz and Sobolev inequalities to bound the nonlinear potential of H with

$$\|u_\alpha u_\beta\|_2^2\leqslant\tfrac{1}{2}\|u_\alpha\|_4^4+\tfrac{1}{2}\|u_\beta\|_4^4\leqslant\frac{C}{2}\|\vec{\nabla}u_\alpha\|_2^2N_\alpha+\frac{C}{2}\|\vec{\nabla}u_\beta\|_2^2N_\beta, \tag{34}$$

which yields

$$H\geqslant\sum_{\alpha=1}^n\|\vec{\nabla}u_\alpha\|_2^2\left(1-\sum_{\beta=1}^n\Lambda_{\alpha\beta}\frac{N_\alpha}{N_c}\right), \tag{35}$$

after making use of the best constant $C_{\text{best}}=2/N_c$ [30] together with the symmetry $\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}$. From the above estimate, we deduce that initial states with energy $H<0$ promote the collapse of *all* the wave components since the total virial integral consists of the direct sum of positive integrals $I_\alpha(t)=N^{-1}\int(\vec{x}-\langle\vec{x}\rangle)^2|u_\alpha|^2d\vec{x}$, implying $I_\alpha(t)\to0$ for every $\alpha=1,...,n$ in the limit $I(t)\to0$. Thus, for a total center of mass being the origin of coordinates, this property implies in turn that each wave component blows up with $\|\vec{\nabla}u_\alpha\|_2\to+\infty$ by virtue of the inequality (15). Self-consistently with Eq. (35), promoting wave collapse requires *a priori* that the partial norms must be above a certain threshold, namely,

$$N_\alpha>\frac{N_c}{\sum_\beta\Lambda_{\alpha\beta}}\quad(\beta=1,...,n). \tag{36}$$

In the case of two coupled identical waves, we simply get $N_1=N_2>N_c'=N_c/(\Lambda_{11}+\Lambda_{12})$, and special interaction regimes can be expected, for which waves amalgamate into one collapsing lobe as their total mean square radius, including their mutual separation distance, decreases to zero in finite time.

To illustrate the above results, let us consider in-phase Gaussian waves with initial shapes

$$u_\alpha(\vec{x},0)=\sqrt{\frac{N_\alpha}{\pi\rho_\alpha^2}}\exp\left[-\frac{[\vec{x}-\langle\vec{x}_\alpha(0)\rangle]^2}{2\rho_\alpha^2}\right] \tag{37}$$

and exhibiting no initial divergence $[\partial_tI(0)=0]$ and no initial velocity $[\partial_t\langle\vec{x}_\alpha(0)\rangle=\vec{0}]$. For such initial envelopes, H expands as

$$H=\sum_{\alpha=1}^n\frac{N_\alpha}{\rho_\alpha^2}\left[1-\sum_{\beta=1}^n\frac{\rho_\alpha^2\Lambda_{\alpha\beta}N_\beta}{2\pi(\rho_\alpha^2+\rho_\beta^2)}e^{-\delta_{\alpha\beta}^2(0)/(\rho_\alpha^2+\rho_\beta^2)}\right], \tag{38}$$

with $\delta_{\alpha\beta}(0)\equiv|\langle\vec{x}_\alpha(0)\rangle-\langle\vec{x}_\beta(0)\rangle|$. Assuming now that the waves possess identical transverse radii ($\rho_\alpha=\rho_\beta\equiv\rho$), H reduces to

$$H=\sum_{\alpha=1}^n\frac{N_\alpha}{\rho^2}\left[1-\sum_\beta\frac{\Lambda_{\alpha\beta}N_\beta}{4\pi}e^{-\delta_{\alpha\beta}^2(0)/2\rho^2}\right]. \tag{39}$$

By using $(N_\alpha-N_\beta)^2\geqslant0$, we deduce that each wave composing a negative-energy state will self-focus provided their power exceeds the critical threshold

$$N_\alpha(\delta\neq0)>\frac{4\pi}{\sum_\beta\Lambda_{\alpha\beta}e^{-\delta_{\alpha\beta}^2(0)/2\rho^2}}, \tag{40}$$

which is always larger than the bound in Eq. (36). Note that the value of the bound from below in Eq. (40) decreases all the more as the mutual separation distance $\delta_{\alpha\beta}(0)$ is small. In particular, superimposed waves with $\delta_{\alpha\beta}(0)=0$ will together collapse if and only if their partial powers exceed the threshold value

$$N_\alpha(\delta=0)>N_c^0\equiv\frac{4\pi}{\sum_\beta\Lambda_{\alpha\beta}}>\frac{N_c}{\sum_\beta\Lambda_{\alpha\beta}}. \tag{41}$$

On the other hand, keeping in mind that $\delta_{\alpha\alpha}(0)=\delta_{\beta\beta}(0)=0$, one can observe that in the limit of well-separated structures $[\delta_{\alpha\beta}(0)\to+\infty$ for $\alpha\neq\beta]$ H is given by the sum of "free" Hamiltonians $H=\Sigma_\alpha H_\alpha=\Sigma_\alpha(N_\alpha/\rho^2)(1-\Lambda_{\alpha\alpha}N_\alpha/4\pi)$, from which we recover the constraint on collapse thresholds $N_\alpha>N_c^f\equiv N_{\text{crit}}/\Lambda_{\alpha\alpha}$ for isolated Gaussians.

Now we denote by $\delta(0)$ the initial distance separating the centroids between two neighboring waves; thus $\delta(0)\equiv\delta_{\alpha\beta}(0)$ for $\alpha\neq\beta$. We search for a critical distance δ_c such that for $\delta(0)>\delta_c$ the waves can be expected to evolve independently of each other, due to the exponential decrease of the interaction term of H. To this aim, we formally rewrite this integral of motion as $H=\Sigma_\alpha H_\alpha+H_{\text{int}}(\delta)=H(\delta=0)+\Delta H(\delta)$. We conjecture that a strong interaction between waves will take place if $\delta(0)$ is such that $|H(0)|\geqslant|\Delta H(\delta(0))|$. Conversely, waves will behave with a negligible correlation if the initial separation distance $\delta(0)$ satisfies $|\Sigma_\alpha H_\alpha|\geqslant|H_{\text{int}}(\delta(0))|$. The critical separation distance below which waves can interact is then given by the zeros of H and it is determined from the general estimate

$$\sum_{\alpha\neq\beta}\Lambda_{\alpha\beta}N_\alpha N_\beta e^{-\delta_c^2/2\rho^2}=\sum_\alpha N_\alpha|4\pi-\Lambda_{\alpha\alpha}N_\alpha|. \tag{42}$$

For two waves δ_c thus reads

$$\delta_c=\rho\left[2\ln\left|\frac{2N_1N_2\Lambda_{12}}{\sum_{\alpha=1,2}N_\alpha(4\pi-\Lambda_{\alpha\alpha}N_\alpha)}\right|\right]^{1/2}, \tag{43}$$

with $\Lambda_{12}=\Lambda_{21}$. For equal waves with $\Lambda_{11}=\Lambda_{22}$, the expression for δ_c simplifies into

$$\delta_c=\rho\left[2\ln\left|\frac{N\Lambda_{12}/\Lambda_{11}}{2N_c^f-N}\right|\right]^{1/2}, \tag{44}$$

with $N=N_1+N_2=2N_\alpha$ and $2N_c^f=2N_{\rm crit}/\Lambda_{11}$. This result agrees with McKinstrie and Russel's result [up to a normalization factor $\frac{1}{2}$ in front of Laplacians of Eq. (26); see Eq. (12) in Ref. [24]], as $2N_c^f$ is equal to $2N_{\rm crit}/\Lambda_{11}=6N_c^0$, where $N_c^0=N_{\rm crit}/(\Lambda_{11}+\Lambda_{12})$ is the power threshold that each of two superimposed waves must exceed to provoke the collapse of the whole wave packet. Choosing $\Lambda_{11}=1/8$ and $\Lambda_{12}=1/4$, one thus gets $N_c^f=32\pi$ and $N_c^0=32\pi/3$. In other situations involving $\Lambda_{11}>\Lambda_{12}$, a nontrivial critical separation distance δ_c exists for high-power waves ($N>2N_c^f$) as long as $N_1=N_2$ lies below the finite limit $N_{\max}=4\pi/(\Lambda_{11}-\Lambda_{12})$ only, whereas $N_{\max}=+\infty$ when $\Lambda_{11}\leqslant\Lambda_{12}$. From the above results and employing the virial properties of the NLS equations, we can then deduce three characteristic regimes for the wave evolution.

(i) If $N_1=N_2<N_c^0$, for any value of $\delta(0)$, the waves contain too a weak power to maintain their localized shapes and they spread out asymptotically with a positive Hamiltonian.

(ii) If $N_c^0<N_1=N_2<N_c^f$, both waves can merge, but they asymptotically disperse with $H>0$ whenever $\delta(0)>\delta_c$. In the opposite situation $\delta(0)\leqslant\delta_c$, wave components fuse and form a self-focusing central lobe with $H<0$ and $I(t)\rightarrow 0$.

(iii) If $N_c^f<N_1=N_2<N_{\max}$, both waves, each having $H_\alpha<0$, generally self-focus independently with an individual power already exceeding the self-focusing threshold for the collapse of a single Gaussian wave. However, if they are only separated initially with $\delta(0)<\delta_c$, they may still amalgamate into a central lobe that will collapse in finite time.

The previous analysis performed for two waves can easily be extended to the mutual interaction of a larger, but finite, number of waves between their nearest neighbors (finiteness here prevents an infinite number of beams from mutually balancing their interaction potentials). For identical waves with $N_1=N_2=\cdots=N_n$ and equally separated by the same distance $\delta(0)$ on a line, the interacting term of H [Eq. (39)] containing the exponentially decreasing contribution expands as

$$\begin{aligned}\frac{1}{4\pi}\sum_{\beta=1,\ldots,n\neq\alpha}\Lambda_{\alpha\beta}N_\beta e^{-\delta^2_{\alpha\beta}(0)/2\rho^2}&=\frac{1}{4\pi}\,[\Lambda_{\alpha1}N_1e^{-\delta^2_{\alpha1}(0)/2\rho^2}\\&\quad+\Lambda_{\alpha2}N_2e^{-\delta^2_{\alpha2}(0)/2\rho^2}+\cdots+\Lambda_{\alpha n}N_ne^{-\delta^2_{\alpha n}(0)/2\rho^2}],\end{aligned} \tag{45}$$

with $\delta^2_{\alpha\beta}(0)\equiv[(\alpha-\beta)\delta(0)]^2$. Let us thus assume that the αth wave refers neither to the first wave nor the nth one. For symmetric nonlinear coefficients $\Lambda_{\alpha\alpha}=\Lambda_{\beta\beta}$ and $\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}$ ($\alpha\neq\beta$) with comparable magnitude ($\Lambda_{\alpha\alpha+1}=\Lambda_{12}$), we get

$$\begin{aligned}\frac{1}{4\pi}\sum_{\beta\neq\alpha}\Lambda_{\alpha\beta}N_\beta e^{-\delta^2_{\alpha\beta}(0)/2\rho^2}&\simeq\frac{1}{4\pi}\,[2\Lambda_{12}N_1e^{-\delta^2(0)/2\rho^2}]\\&\quad+O(\Lambda_{13}N_1e^{-2\delta^2(0)/\rho^2})\end{aligned} \tag{46}$$

for well-separated waves initially satisfying $\delta(0)>2\rho$ (see, e.g., Ref. [10]). Thus nonlinear interaction regimes between coupled self-focusing beams can be reduced to the problem of the interaction of three waves on a line, i.e., the αth wave with two lateral ones. The basin of attraction for mutual entrainment of one light spot surrounded by two equal others on a line then decreases when $\delta(0)$ is above a critical value δ_c whose expression is the same as in Eq. (44) with Λ_{12} replaced by $2\Lambda_{12}$. Similarly, it could be checked that the interaction regimes between light pulses distributed on a plane lattice reduce to those of one αth wave with its four nearest neighbors. For n^2 spots regularly spaced on a lattice in the transverse plane, the interaction between the αth wave with respect to its four nearest neighbors increases by a factor 2 compared to spots spaced on a line, so that the value of δ_c is the same as in Eq. (44), but with a crossed nonlinearity coefficient Λ_{12} multiplied by 4. From these arguments, we easily infer that the critical distance below which waves can coalesce increases with the number of waves and their geometrical space.

Besides, the formation of a central lobe and its ultimate self-focusing fully develops in the medium before the first zero $t_c^{\max}$ of the total virial integral $I(t)$. After this instant, the set composed by the n NLS solutions u_α ($\alpha=1,\ldots,n$) can no longer exist. In the two-dimensional case, Eq. (33) leads to

$$I(t)=\frac{4Ht^2}{N}+[\partial_tI(0)]t+I(0), \tag{47}$$

with H defined by Eq. (39). Assuming then $H<0$ for Gaussian waves with no initial divergence, the maximal time for the blow-up of all waves is given by

$$\begin{aligned}t_c^{\max}(\delta)&\\&=\left[\frac{\pi NI(0)\rho^2}{\sum_\alpha N_\alpha\left\{\sum_\beta\Lambda_{\alpha\beta}N_\beta\exp\{-\delta^2_{\alpha\beta}(0)/2\rho^2\}-4\pi\right\}}\right]^{1/2}\\&<+\infty.\end{aligned} \tag{48}$$

With $I(0)=\rho^2+\Sigma_\alpha N_\alpha\langle\vec{x}_\alpha(0)\rangle^2/N$, we finally deduce that the maximum collapse moment increases with the initial distance separating the coordinate of the wave centroid from the origin and when the separation distance between waves increases [note that for identical and symmetrical waves one has $|\langle\vec{x}_\alpha(0)\rangle|=\delta_{\alpha\beta}(0)/2$]. Thus the collapse (or self-focusing) time may be "tuned" for experimental convenience by fixing in a appropriate way: (i) the number of light spots, (ii) their respective incident powers, and (iii) their location in space. A plot of δ_c versus N and another one illustrating $t_c^{\max}$ versus $\delta(0)$ and $N=nN_\alpha$ have been given in Fig. 1 for different powers in the case of two waves ($n=2$) with

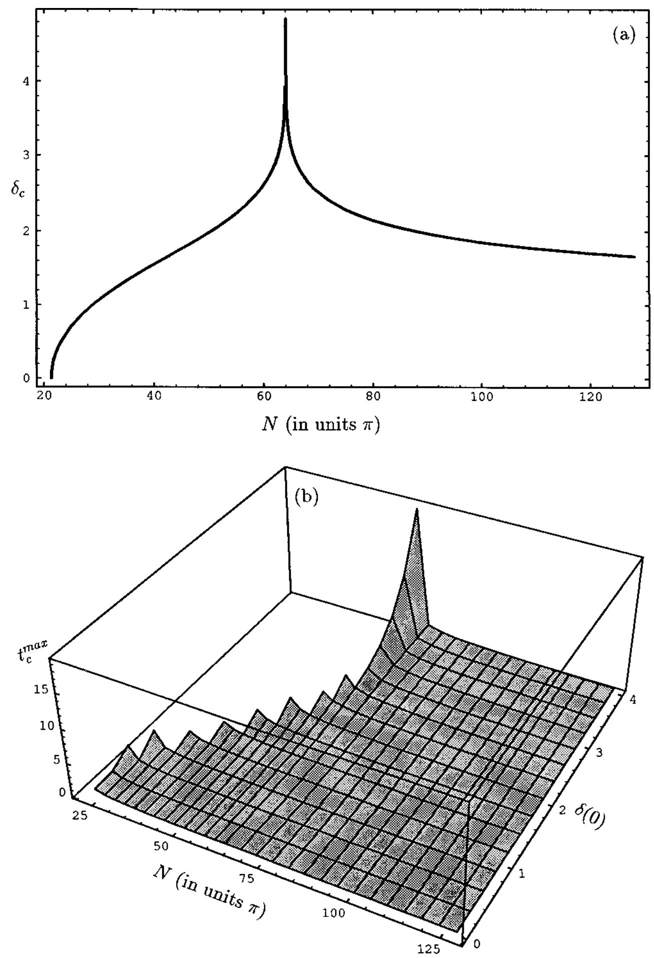

FIG. 1. (a) 2D plot of δ_c [Eq. (44)] versus $N=2N_\alpha$ and (b) 3D plot of $t_c^{\max}$ [Eq. (48)] versus N and $\delta(0)$ for two equal waves symmetrically located from the origin. The power is expressed in units of π and the nonlinearity coefficients satisfy $\Lambda_{12}/\Lambda_{11}=2$. Two distinct ranges of power allowing for the existence of δ_c delimit the interaction regimes (ii) and (iii), namely, $2N_0^c=64\pi/3<N<2N_0^f=64\pi$ and $N>2N_0^f=64\pi$. From (b), $t_c^{\max}$ can be seen to decrease with the wave power and to increase with the initial separation distance between wave centroids.

identical individual powers. Even though the estimate (48) of the collapse time consists only of the maximal instant for blow-up, it is finally worth noticing that an empirical estimate for the time taken by the coalescence of waves itself may be yielded by $\Delta t^{\text{coalesc}}\approx t_c^{\max}(\delta)-t_c^{\max}(\delta=0)$, when one keeps in mind that overlapped waves should promote the fastest collapse process.

III. VARIATIONAL APPROACH TO INTERACTING WAVES

To investigate the dynamics of mutually interacting beams, we perform a variational approach (the so-called average Lagrangian method) allowing us to depict the main tendencies of coupled light pulses. We use the property following which Eq. (26) derives from the Lagrange equations $\delta L/\delta u_\alpha^*=\delta L/\delta u_\alpha=0$, where L is the Lagrangian integral

$$L=\frac{i}{2}\sum_\alpha \int (u_\alpha^*\partial_t u_\alpha - u_\alpha \partial_t u_\alpha^*)d\vec{x}-H, \qquad (49)$$

in which the Hamiltonian H has already been defined in Eq. (27) for symmetric nonlinearity coefficients $\Lambda_{\alpha\beta}=\Lambda_{\beta\alpha}$. The variational approach, constructed from a Rayleigh-Ritz principle, assures *a priori* that the wave dynamics may be described in terms of a finite number of time-dependent parameters entering a trial function appropriately chosen to fit the true solutions of the coupled NLS equations (26), obtained, for instance, from their numerical integration. This trial function is elaborated from a test function, standardly chosen among Gaussians or sech functions, which ensures a well-localized shape for the wave forms. It is now well known that for a single wave governed by the cubic NLS equation, sech test functions are more suitable than Gaussians in the sense that, e.g., with a nonlinearity coefficient equal to unity, the critical threshold power for self-focusing computed from sech functions ($N_c^{\text{sech}}=11.72$) is closer to the minimum value $N_c=11.68$ than when it is computed from Gaussians ($N_c^{\text{Gauss}}=N_{\text{crit}}=4\pi$) [19,31]. This discrepancy, however, consists of a narrow margin of error that we can further omit because the dynamical aspects such as self-focusing or wave spreading sorted out from Gaussian functions are qualitatively the same as for sech functions. Also, by starting initially with Gaussian optical wave packets as they are often introduced in experimental setups, it seems more natural to select a Gaussian test function fitting the optical pulse at least at $t=0$. Nevertheless, sech functions will be discussed at the end of this section for modeling 2D collapsing wave packets. In the 1D case where no collapse occurs, application of the variational approach with sech test functions has been performed by Anderson and Lisak [21] and then by Ueda and Kath [22] to describe an incoherent two-soliton interaction and the nonlinear coupling of two NLS waves in optical fibers, respectively.

A. General derivation of the variational equations for coupled waves

We first consider a test function with a general form ϕ_α depending on the rescaled spatial variables $\vec{\xi}_\alpha\equiv[\vec{x}-\langle\vec{x}_\alpha(t)\rangle]/a_\alpha(t)$, where $\langle\vec{x}_\alpha(t)\rangle$ denotes the centroid of the αth wave and $a_\alpha(t)$ its typical time-varying radius. From Eq. (26) it is easy to derive the continuity equation for the power conservation of this wave:

$$\partial_t|u_\alpha|^2=-2\vec{\nabla}\cdot[|u_\alpha|^2\vec{\nabla}\ \arg(u_\alpha)]. \qquad (50)$$

This relation, assumed to keep a covariant form after substituting a trial function with a self-similar ($\vec{\xi}_\alpha$-dependent) shape [28], suggests the choice of u_α with a real test function ϕ_α in the form

$$u_\alpha(\vec{x},t)=\frac{1}{[a_\alpha(t)]^{D/2}}\,\phi_\alpha\!\left(\frac{\vec{x}-\langle\vec{x}_\alpha(t)\rangle}{a_\alpha(t)}\right)\times\exp\!\left[i\,\frac{\dot{a}_\alpha(t)}{4a_\alpha(t)}\,[\vec{x}-\langle\vec{x}_\alpha(t)\rangle]^2+\frac{i}{2}\,\langle\dot{\vec{x}}_\alpha(t)\rangle\cdot[\vec{x}-\langle\vec{x}_\alpha(t)\rangle]+i\zeta_\alpha(t)\right], \qquad (51)$$

where $\zeta_\alpha(t)$ denotes an arbitrary time-dependent phase factor. In the substitution (51), the space-dependent phase is then necessary to balance self-consistently the additional contributions coming from the time derivatives of $a_\alpha(t)$ in the self-similarly transformed version of Eq. (50). Generalizing expression (51), we now perform the standard average Lagrangian procedure by employing the more general solution

$$u_\alpha(\vec{x},t)=\frac{1}{\sqrt{J_\alpha(t)}}\,\phi_\alpha\left(\frac{\vec{x}-\langle\vec{x}_\alpha(t)\rangle}{a_\alpha(t)}\right)\exp\Big\{i\theta_\alpha(t)[\vec{x}-\langle\vec{x}_\alpha(t)\rangle]^2+\frac{i}{2}\,\vec{\psi}_\alpha(t)\cdot[\vec{x}-\langle\vec{x}_\alpha(t)\rangle]+i\zeta_\alpha(t)\Big\},\tag{52}$$

where $\theta_\alpha(t)$ accounts for the quadratic chirp parameter and the vector $\vec{\psi}_\alpha(t)$ is related to the velocity of the centroid displacement $\langle\vec{x}_\alpha(t)\rangle$ for the αth wave. $J_\alpha(t)$ is a real amplitude factor such that $J_\alpha(t)\to 0$ when wave collapse develops in finite time. Plugging Eq. (52) into the Lagrangian (49), we find that this integral explicitly reads

$$\begin{aligned}L=\sum_{\alpha=1}^{n}\Bigg\{\frac{a_\alpha^D}{J_\alpha}\Bigg[&V_\alpha(-\dot{\theta}_\alpha a_\alpha^2-4\theta_\alpha^2 a_\alpha^2)\\&+\vec{C}_\alpha\cdot(2\langle\dot{\vec{x}}_\alpha\rangle a_\alpha\theta_\alpha-\tfrac{1}{2}a_\alpha\dot{\vec{\psi}}_\alpha-2\theta_\alpha a_\alpha\vec{\psi}_\alpha)\\&+M_\alpha\left(\frac{\langle\dot{\vec{x}}_\alpha\rangle}{2}\cdot\vec{\psi}_\alpha-\dot{\zeta}_\alpha-\frac{\vec{\psi}_\alpha^2}{4}\right)-\frac{\overline{D}_\alpha}{a_\alpha^2}\Bigg]\Bigg\}\\&+\sum_{\alpha,\beta=1}^{n}\frac{\Lambda_{\alpha\beta}}{2J_\alpha J_\beta}\,a_\alpha^D W_{\alpha\beta}(\vec{\delta}_{\alpha\beta},a_\alpha,a_\beta),\end{aligned}\tag{53}$$

with $\vec{\delta}_{\alpha\beta}(t)\equiv\langle\vec{x}_\alpha(t)\rangle-\langle\vec{x}_\beta(t)\rangle$. This formulation of the Lagrangian involves the coefficients

$$\begin{aligned}&V_\alpha\equiv\int\xi_\alpha^2|\phi_\alpha|^2d\vec{\xi}_\alpha,\quad \vec{C}_\alpha\equiv\int\vec{\xi}_\alpha|\phi_\alpha|^2d\vec{\xi}_\alpha,\\&M_\alpha\equiv\int|\phi_\alpha|^2d\vec{\xi}_\alpha,\\&\overline{D}_\alpha\equiv\int|\vec{\nabla}_{\xi_\alpha}\phi_\alpha|^2d\vec{\xi}_\alpha,\quad W_{\alpha\beta}(\vec{\delta}_{\alpha\beta},a_\alpha,a_\beta)\\&\equiv\int|\phi_\alpha(\vec{\xi}_\alpha)|^2\left|\phi_\beta\left(\frac{a_\alpha\vec{\xi}_\alpha+\vec{\delta}_{\alpha\beta}}{a_\beta}\right)\right|^2d\vec{\xi}_\alpha,\end{aligned}\tag{54}$$

with $\phi_\alpha=\phi_\alpha^*$. Here the integrals are taken in the whole space range $(-\infty<\vec{\xi}_\alpha<+\infty)$ and $\vec{\nabla}_{\xi_\alpha}$ means $\vec{\nabla}_{\xi_\alpha}=\vec{\nabla}(\vec{x}\to\vec{\xi}_\alpha)$. For the sake of simplicity, we assume an even test function ϕ_α such that the vector $\vec{C}_\alpha$ reduces to zero. The variational equations obtained by performing the functional derivatives of L with respect to the entire set of time-dependent parameters $(J_\alpha,\theta_\alpha,\langle\dot{\vec{x}}_\alpha\rangle,a_\alpha,\zeta_\alpha,\vec{\psi}_\alpha)$ constitute a dynamical system describing the global tendencies of the temporal evolution of the coupled waves. The simplest relation is derived from $\delta L/\delta\zeta_\alpha=0$: It restores the conservation of power with $a_\alpha^D(t)/J_\alpha(t)=a_\alpha^D(0)/J_\alpha(0)$, for which, without loss of generality, we can choose $a_\alpha^D(0)=J_\alpha(0)$, implying thereby $a_\alpha^D(t)=J_\alpha(t)$. Keeping in mind that each wave conserves its individual power, we then get $N_\alpha=M_\alpha a_\alpha^D/J_\alpha=M_\alpha$. The remaining relations are derived from L with respect to $(\theta_\alpha,\langle\dot{\vec{x}}_\alpha\rangle,\vec{\psi}_\alpha,a_\alpha)$ and expand after straightforward calculations as

$$\begin{aligned}&\frac{\delta L}{\delta\theta_\alpha}=0\Rightarrow\theta_\alpha=\frac{\dot{a}_\alpha}{4a_\alpha},\\&\frac{\delta L}{\delta\vec{\psi}_\alpha}=0\Rightarrow\vec{\psi}_\alpha=\langle\dot{\vec{x}}_\alpha\rangle,\end{aligned}\tag{55}$$

$$\frac{\delta L}{\delta\langle\vec{x}_\alpha\rangle}=\vec{0}\Rightarrow\langle\ddot{\vec{x}}_\alpha\rangle=-\sum_{\beta\neq\alpha,\beta=1}^{n}\frac{\Lambda_{\alpha\beta}}{M_\alpha a_\beta^D}\,\partial_{\langle\vec{x}_\alpha\rangle}W_{\alpha\beta}(\vec{\delta}_{\alpha\beta},a_\alpha,a_\beta),\tag{56}$$

$$\frac{\delta L}{\delta a_\alpha}=0\Rightarrow\frac{V_\alpha}{2}\,\ddot{a}_\alpha=\frac{2\overline{D}_\alpha}{a_\alpha^3}-\frac{D\Lambda_{\alpha\alpha}}{2a_\alpha^{D+1}}\,W_{\alpha\beta}(\vec{0},a_\alpha=a_\beta)+\sum_{\beta\neq\alpha,\beta=1}^{n}\frac{\Lambda_{\alpha\beta}}{2}\,\partial_{a_\alpha}[a_\beta^{-D}W_{\alpha\beta}(\vec{\delta}_{\alpha\beta},a_\alpha,a_\beta)],\tag{57}$$

where $\partial_{a_\alpha}\equiv\partial/\partial a_\alpha$. Although improper, the notation $\partial_{\langle\vec{x}_\alpha\rangle}$ means a derivative with respect to the central position of the αth pulse, affected by its unitary orientation vector $\vec{e}_\alpha$ (strictly speaking, one should introduce an auxiliary angular variable ω_α entering the definition $\langle\vec{x}_\alpha\rangle\equiv|\langle\vec{x}_\alpha\rangle|(\cos\omega_\alpha,\sin\omega_\alpha)$ and derive the Euler-Lagrange equations with respect to this cyclic variable too. However, for technical convenience, we omit it and treat the vector $\langle\vec{x}_\alpha\rangle$ as a canonical collective coordinate in our Lagrangian approach). Taking into account the inversion symmetry between the αth and βth waves, $\vec{\delta}_{\alpha\beta}=-\vec{\delta}_{\beta\alpha}$, it is then easy to find again the conservation law for the total wave momentum linked to the total wave centroid by the relation $\partial_t^2\langle\vec{x}\rangle=\partial_t^2\Sigma_{\alpha=1}^n N_\alpha\langle\vec{x}_\alpha\rangle/N=\partial_t\vec{P}/N=0$ since

$$\sum_{\alpha=1}^{n}M_\alpha\langle\ddot{\vec{x}}_\alpha\rangle=\sum_{\alpha\neq\beta,\alpha,\beta=1}^{n}\frac{\Lambda_{\alpha\beta}}{a_\beta^D}\,[\partial_{\langle\vec{x}_\alpha\rangle}W_{\alpha\beta}+\partial_{\langle\vec{x}_\beta\rangle}W_{\alpha\beta}]=\vec{0}.\tag{58}$$

In addition, the variational equations restore the conservation of the Hamiltonian, which now is expressed as $H\equiv H^{\mathrm{free}}+H^{\mathrm{ext}}$ with

$$H^{\rm free} \equiv \sum_{\alpha=1}^{n} \left[\frac{\overline{D}_\alpha}{a_\alpha^2} + \frac{\dot{a}_\alpha^2 V_\alpha}{4} - \frac{\Lambda_{\alpha\alpha}}{2a_\alpha^D} W_{\alpha\alpha}(\vec{0}, a_\alpha = a_\beta) \right], \tag{59}$$

$$H^{\rm ext} \equiv \sum_{\alpha=1}^{n} \frac{M_\alpha}{4} \langle \dot{\vec{x}}_\alpha \rangle^2 - \sum_{\alpha\neq\beta,\alpha,\beta=1}^{n} \frac{\Lambda_{\alpha\beta}}{2a_\beta^D} W_{\alpha\beta}(\vec{\delta}_{\alpha\beta}, a_\alpha, a_\beta). \tag{60}$$

$H^{\rm ext}$ contains the kinetics of the pulses whose centroids move with the velocity $\langle \dot{\vec{x}}_\alpha \rangle$ and the potential energy due to their mutual coupling, denoted by $H^{\rm int} \equiv -\Sigma_{\alpha\neq\beta} \Lambda_{\alpha\beta} W_{\alpha\beta}/2a_\beta^D$. From the above expression, we can distinguish different dynamics of the pulses following the sign of H. Starting with initially unchirped $[\dot{a}_\alpha(0)=0]$ pulses and considering the case $D=2$, waves with a zero initial velocity in their centroids may collapse individually if $H^{\rm free}<0$ and with a negligible interaction term measured through $W_{\alpha\beta}$. Alternatively, even if $H^{\rm free}$ is positive, but whenever $H^{\rm int}$ still guarantees $H<0$, pulses can attract each other and form a central lobe that will collapse in finite time. On the contrary, waves with $H>0$ will ultimately disperse, after fusing or not. Between both of these behaviors, we have the boundary case $H=0$, from which we can define an ''escape'' velocity of collapsing wave forms. This escape velocity corresponds to the minimum speed of centroids that initially superimposed waves must possess to become detrapped and overcome the collapse. Following this definition, the escape velocity is given for the αth wave by

$$v_{e,\alpha}^2 = \frac{4}{M_\alpha a_\alpha^2(0)} \left[\sum_\beta \frac{\Lambda_{\alpha\beta}}{2} \frac{a_\alpha^2(0)}{a_\beta^2(0)} W_{\alpha\beta}(\vec{0}, a_\alpha, a_\beta) - \overline{D}_\alpha \right]. \tag{61}$$

For two identical Gaussian pulses ($N_\alpha = N_\beta$), its expression simplifies into $v_e^2 = 4[(N_\alpha/N_c^0)-1]$ and it makes sense when pulses form a collapsing bound state with sufficiently high powers $N_\alpha > N_c^0 = 4\pi/(\Lambda_{\alpha\alpha}+\Lambda_{\alpha\beta})$.

B. Interaction of Gaussian pulses

From now on, we investigate the dimensional case $D=2$ suitable for describing a paraxial self-focusing. We make use of the Gaussian test function

$$\phi_\alpha(\vec{\xi}_\alpha) = \sqrt{K_\alpha} e^{-\xi_\alpha^2/2}. \tag{62}$$

This leads in particular to the interaction term for the nonlinear potential of L:

$$W_{\alpha\beta}(\vec{\delta}_{\alpha\beta}, a_\alpha, a_\beta) = \pi K_\alpha K_\beta \frac{a_\beta^2}{a_\alpha^2 + a_\beta^2} e^{-\delta_{\alpha\beta}^2/(a_\alpha^2+a_\beta^2)}, \tag{63}$$

where the separation distance $\delta_{\alpha\beta}$ is now a time-varying function, and the remaining coefficients (54) appearing in the average formulation of L simply read

$$V_\alpha = \overline{D}_\alpha = M_\alpha = \pi K_\alpha, \tag{64}$$

such that $N_\alpha = \pi K_\alpha$. The wave centroids and widths are then found to be governed by the dynamical relations

$$\langle \ddot{\vec{x}}_\alpha \rangle = -\sum_{\beta\neq\alpha} \frac{2\Lambda_{\alpha\beta} N_\beta}{\pi(a_\alpha^2+a_\beta^2)^2} \vec{\delta}_{\alpha\beta} e^{-\delta_{\alpha\beta}^2/(a_\alpha^2+a_\beta^2)}, \tag{65}$$

$$\frac{\ddot{a}_\alpha}{4} = \left(1 - \frac{\Lambda_{\alpha\alpha} N_\alpha}{4\pi}\right) \frac{1}{a_\alpha^3} - \sum_{\beta\neq\alpha} \frac{\Lambda_{\alpha\beta} N_\beta}{2\pi(a_\alpha^2+a_\beta^2)^2} \times a_\alpha \left\{ 1 - \frac{\delta_{\alpha\beta}^2}{(a_\alpha^2+a_\beta^2)} \right\} e^{-\delta_{\alpha\beta}^2/(a_\alpha^2+a_\beta^2)}. \tag{66}$$

Applying Eq. (65) to two neighboring waves, we construct an equation for $\vec{\delta}_{\alpha\beta}(t)$ $(\alpha,\beta=1,2)$:

$$\ddot{\vec{\delta}}_{\alpha\beta} = -\frac{2}{\pi} \sum_{\alpha\neq\beta} \frac{(\Lambda_{\alpha\beta} N_\beta + \Lambda_{\beta\alpha} N_\alpha)}{(a_\alpha^2+a_\beta^2)^2} \vec{\delta}_{\alpha\beta} e^{-\delta_{\alpha\beta}^2/(a_\alpha^2+a_\beta^2)}. \tag{67}$$

Let us now investigate the case of two identical copropagating waves with equal powers $N_1 = N_2$ and undergoing a mutual coupling with symmetric nonlinearity coefficients such as those employed in Ref. [24], i.e., $\Lambda_{11} = \Lambda_{22} = \frac{1}{8}$ and $\Lambda_{12} = \Lambda_{21} = \frac{1}{4}$. In this case, the separation vector evolves as

$$\ddot{\vec{\delta}} = -\frac{8N_1}{\pi(a_1^2+a_2^2)^2} \Lambda_{12} \vec{\delta} \exp\left[-\frac{\delta^2}{a_1^2+a_2^2} \right],$$

$$\vec{\delta}(t) \equiv \langle \vec{x}_1(t) \rangle - \langle \vec{x}_2(t) \rangle, \tag{68}$$

while the wave radii $a_1(t)$ and $a_2(t)$ are governed by analogous dynamical equations: The equation for $a_1(t)$ reads

$$\frac{\ddot{a}_1}{4} = \left(1 - \frac{\Lambda_{11} N_1}{4\pi}\right) \frac{1}{a_1^3} - \frac{\Lambda_{12} N_1}{\pi(a_1^2+a_2^2)^2} \times a_1 \left\{ 1 - \frac{\delta^2}{(a_1^2+a_2^2)} \right\} \exp\left[-\frac{\delta^2}{a_1^2+a_2^2} \right] \tag{69}$$

and the second equation for $a_2(t)$ proceeds from the same equation (69) in which the indices 1 and 2 have to be reversed. So, starting with identical symmetric waves having equal initial conditions with $a_1(0) = a_2(0)$ and $\dot{a}_1(0) = \dot{a}_2(0)$, one obviously has $a_1(t) = a_2(t)$. When such waves are initially superimposed with $\vec{\delta}(0) = \vec{0}$ and $\dot{\vec{\delta}}(0) = \vec{0}$, they stay mutually overlapped at every later time and their radii are governed by the relation

$$\frac{\ddot{a}_1}{4} = \left(1 - \frac{\Lambda_{11}+\Lambda_{12}}{4\pi} N_1\right) \frac{1}{a_1^3}, \quad a_1 = a_2. \tag{70}$$

Thus initially superimposed waves collapse in finite time provided $N_1 = N_2 > N_c^0 = N_{\rm crit}/(\Lambda_{11}+\Lambda_{12}) = 8\times 4\pi/3$, in accordance with Eq. (41). In the opposite limit $|\vec{\delta}(0)| \to +\infty$ with $|\partial_t \vec{\delta}(0)| = 0$, Eq. (68) still yields $|\vec{\delta}(t)| = |\vec{\delta}(0)| = +\infty$ at every time and $a_1(t)$ is governed by

$$\frac{\ddot{a}_1}{4} = \left(1 - \frac{\Lambda_{11} N_1}{4\pi}\right) \frac{1}{a_1^3}, \quad a_1 = a_2, \tag{71}$$

which restores the usual self-focusing threshold for one Gaussian wave: $N_2 = N_1 > N_c^f \equiv N_{\rm crit}/\Lambda_{11} = 8\times 4\pi$. For equal waves such as $N=N_1+N_2=2N_1$, we recall that the critical distance beyond which well-separated waves can behave independently of each other is given by Eq. (44) with $\Lambda_{12}=2\Lambda_{11}$:

$$\delta_c=\rho\left[2\ln\left|\frac{2N}{2N_c^f-N}\right|\right]^{1/2},\quad \rho=a_1(0)=a_2(0). \tag{72}$$

With $\delta(0)\leqslant\delta_c$, waves are expected to mutually attract and merge into one central lobe if $N_1=N_2$ satisfies

$$N_c^0=\frac{32\pi}{3}<N_1=N_2<N_c^f=32\pi. \tag{73}$$

More precisely, for $\delta(0)<\delta_c$, the Hamiltonian is close to that for superimposed waves, which is negative. So, in this range of power values, Gaussian waves should merge and fuse into a central lobe that is condemned to collapse in finite time with a resultant power exceeding the critical threshold for self-focusing. In addition, due to the presence of the absolute value in the estimate (72), the existence of a finite separation distance, below which two equal waves with $N_1=N_2>N_c^f$ must fuse and self-focus, makes sense, even though those regimes favor individual collapses for waves containing separately a power larger than the critical threshold N_c^f. From these possibilities, we emphasize the three typical interaction regimes between both waves, as they have been introduced in Sec. II, namely, (i) whatever $\delta(0)$ may be, the two waves spread out when their individual power is below N_c^0; (ii) for $N_c^0=32\pi/3<N_1=N_2<N_c^f=32\pi$, the waves merge when $\delta(0)$ is relatively close to δ_c and form a dispersing structure if initially $\delta(0)>\delta_c$ or, conversely, they fuse into a central lobe that collapses in finite time if $\delta(0)\leqslant\delta_c$; and (iii) for $N_1=N_2>N_c^f=32\pi$, the two waves generally collapse in finite time individually, except if their initial separation distance is smaller than δ_c, in which case they may amalgamate until forming a collapsing structure.

In this last situation, coalescence of waves can clearly develop if they are separable initially, i.e., if their mutual separation distance is initially at least larger than two times their radii with $\delta(0)\geqslant 2\rho=2a_1(0)=2a_2(0)$. The two maxima of the waves are then well separable [10]. Note that the double condition $\delta_c\geqslant\delta(0)>\delta_N\equiv C\rho$ with $C=2$ implies a bound from above for $N_1=N_2$ when assuming *a priori* $N_\alpha>N_c^f$, that is, $N_c^f<N_\alpha<N_{\sup}=N_c^f/(1-2e^{-C^2/2})=1.371N_c^f$. Conversely, this constraint introduces a bound from below in the opposite situation $N_\alpha<N_c^f$, that is, $N_c^f>N_\alpha>N_{\inf}=0.787N_c^f$. These intervals become narrower when increasing the value of C: Choosing $C=2.2$ as in [24] yields $N_{\sup}=1.216N_c^f$ and $N_{\inf}=0.85N_c^f$. To illustrate the different interaction regimes for two waves, we have numerically integrated Eqs. (68) and (69) and plotted the corresponding curves for the vector $\vec{\delta}(t)$ and the radius $a_1(t)=a_2(t)$ in Fig. 2, starting with $\rho=a_1(0)=a_2(0)=1$, $\dot{a}_1(0)=\dot{a}_2(0)=0$, and $\partial_t\vec{\delta}(0)=\vec{0}$. (i) For $N_1=N_2=10\pi$, both waves spread out monotonically [Fig. 2(a)]. (ii) For $N_c^0<N_1=N_2=30\pi<N_c^f$, the waves merge and form a dispersing wave form provided $\delta(0)>\delta_c\simeq 2.61\rho$ [Fig. 2(b)], whereas they self-focus into one central lobe in the opposite limit $\delta(0)\leqslant\delta_c\simeq 2.61\rho$ [Fig. 2(c)]. This value of δ_c, exactly reading $\delta_c=2.608\,14$ and computed with Eq. (72) for $\rho=1$, is in excellent agreement with the one identified numerically from the variational equations: $\delta_c^{\mathrm{var}}=2.6083$. Finally, the waves self-focus separately for $N_1=N_2=35\pi>N_c^f$ provided $\delta(0)$ is at least above $\delta_c=2.51$, as expected [Fig. 2(d)]. At very high powers $N_1=N_2\gg N_c^f$, the critical distance δ_c becomes formally independent of $N=N_1+N_2$, with $\delta_c\to(2\ln 2)^{1/2}\rho$ as $N\to+\infty$ and δ_c thus diminishes as N increases from $2N_c^f$ in this power regime. Thus the waves are more difficult to amalgamate when they both possess a power strong enough to promote individual collapses. Conversely, when $N_1=N_2<N_c^f$, δ_c increases with N until infinity as $N\to 2N_c^f$.

We can remark that in the amalgamation regimes characterized by the fusion $[\delta(t)\to 0]$ and collapse $[a_1(t)\to 0]$ of coupled waves, the quantities $\vec{\delta}(t)$ and $a_1(t)$ may have an oscillatory behavior during transient evolutions. Oscillations in $\vec{\delta}(t)$ characterize wave components passing through each other and they have also been detected in the interaction of two incoherent [21] and coherent [32] NLS solitons treated by a similar Rayleigh-Ritz principle. Pulsations in the wave widths can be compared with the steady oscillations developed by soliton solutions to the 1D version of Eq. (26) when, e.g., they are close to become mutually trapped, as discovered in Ref. [22]. Because the widths $a_\alpha(t)$ are driven by the motions of the central positions $\langle\vec{x}_\alpha(t)\rangle$, both these quantities oscillate with analogous periods that are here either modulated or damped when collapse occurs. Oscillations in the separation distance $\delta(t)$ can be understood from the first integral of motion computed with Eq. (68) for two waves with equal amplitude and width $[a_1(t)=a_2(t)]$.

In this configuration let us indeed multiply Eq. (68) by $\dot{\vec{\delta}}$ and combine it with Eq. (69) multiplied by $\dot{a}_1$. By doing so and integrating the resulting equation once in time, we obtain the potential function formulation

$$\tfrac{1}{2}(\dot{\vec{\delta}})^2+2(\dot{a}_1)^2+\Pi(\delta,a_1)=0, \tag{74}$$

$$\begin{aligned}\Pi(\delta,a_1)\equiv&\left(4-\frac{\Lambda_{11}N_1}{\pi}\right)\left(\frac{2}{a_1^2}-\frac{2}{a_{01}^2}\right)\\&-\frac{2N_1}{\pi}\Lambda_{12}\left(\frac{e^{-\delta^2/2a_1^2}}{a_1^2}-\frac{e^{-\delta_0^2/2a_{01}^2}}{a_{01}^2}\right)\\&-\frac{1}{2}(\dot{\vec{\delta}}_0)^2-2(\dot{a}_{01})^2,\end{aligned} \tag{75}$$

with $\vec{\delta}_0\equiv\vec{\delta}(0)$, $\dot{\vec{\delta}}_0\equiv\dot{\vec{\delta}}(0)$, $a_{01}\equiv a_1(0)=\rho$, and $\dot{a}_{01}\equiv\dot{a}_1(0)$. From Eq. (74) it is clear that solutions $\delta(t)\equiv|\vec{\delta}(t)|$ and $a_1(t)$ may exist only if the potential function $\Pi(\delta,a_1)$ is negative. For subcritical powers $N_1<N_{\mathrm{crit}}/(\Lambda_{11}+\Lambda_{12})$ leading to an increasing $a_1(t)$, this condition $\Pi(\delta,a_1)<0$ is systematically satisfied when

$$\frac{1}{2}(\dot{\vec{\delta}}_0)^2+2(\dot{a}_{01})^2\geqslant\frac{2N_1}{\pi}\Lambda_{12}\frac{e^{-\delta_0^2/2a_{01}^2}}{a_{01}^2}. \tag{76}$$

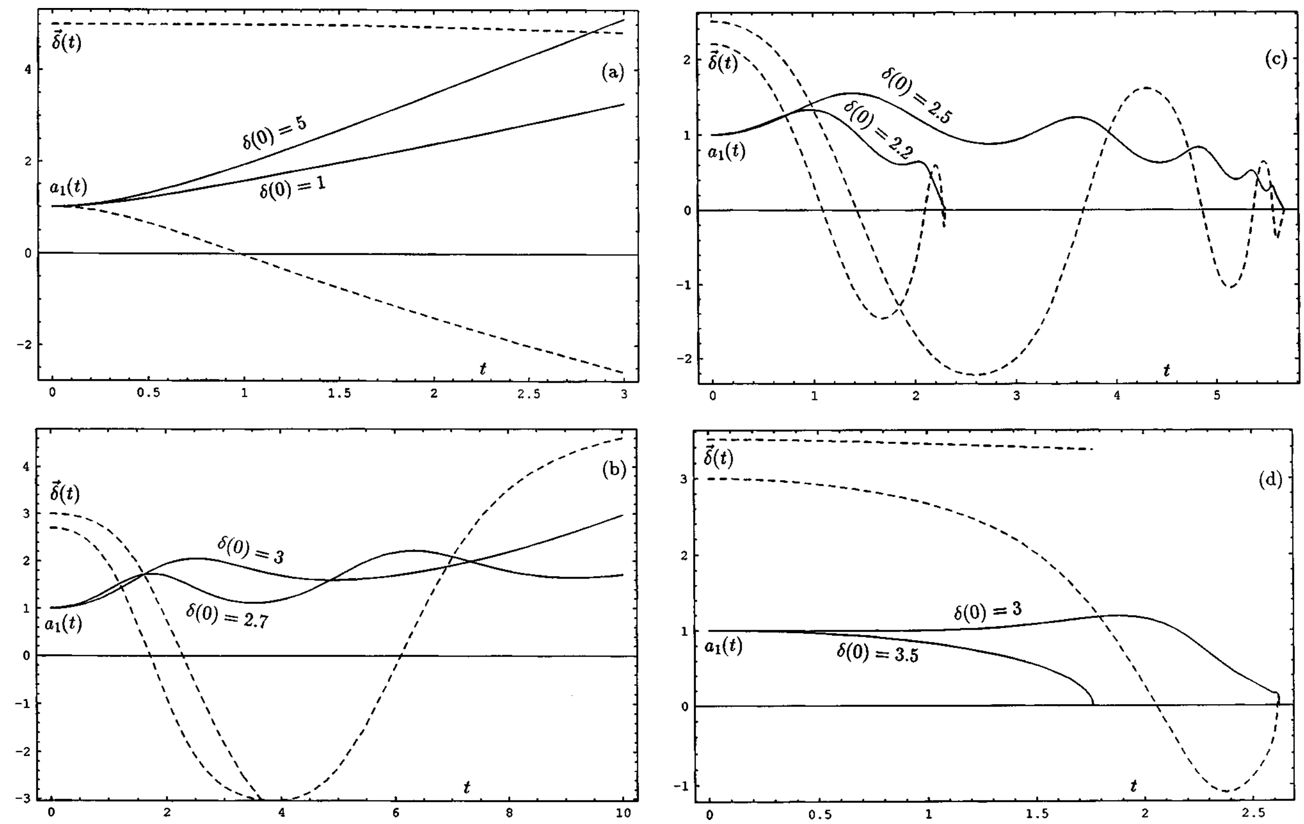

FIG. 2. Wave radius $a_1(t)=a_2(t)$ starting from unity without initial divergence [only $a_1(t)$ is shown as a solid line] and separation vector $\vec{\delta}(t)$ (dashed line) versus time, numerically integrated from the variational equations (68) and (69). By abuse of notation, $\vec{\delta}(t)$ has been kept as a vector, instead of modulus, in order to permit it to become negative. The value of the initial separation distance between centroids, which determines the evolution of the wave radius with $\dot{\delta}(0)=0$, has been indicated around the solid curves. (a) Interaction regime (i) for which the two waves with a weak power ($N_\alpha=10\pi$) monotonically disperse. For $\delta(0)$ far above 5, wave radii were observed to disperse identically. (b) Interaction regime (ii) for which waves with partial power $N_\alpha=30\pi$ merge and spread out asymptotically when their mutual separation distance initially satisfies $\delta(0)>\delta_c\simeq 2.608$. (c) Interaction regime (ii) for which the same waves as in (b) amalgamate into one central self-focusing lobe when $\delta(0)\leqslant\delta_c$. Note the oscillations in the separation vector that are damped to zero. (d) Interaction regime (iii) for which waves with $N_\alpha=35\pi$ collapse individually before their separation distance reaches zero, except when $\delta(0)$ is initially less than a critical value. Numerically, it was observed that this critical distance of separation was $\delta_c\simeq 3.14$, i.e., slightly above the estimate (44) yielding $\delta_c=2.51$ for $N_\alpha=35\pi$.

For the initial data such as $\dot{\vec{\delta}}_0=\vec{0}$ and $\dot{a}_{01}=0$, however, $\Pi(\delta,a_1)$ is negative provided that $\delta(t)$ belongs to the bounded range $\delta(t)\leqslant\delta_*$, with δ_* defined by

$$\delta_*=\sqrt{2}a_1(t)\left\{\ln\left(\frac{4\pi}{N_1\Lambda_{12}}-\frac{\Lambda_{11}}{\Lambda_{12}}\right)^{-1} -\ln\left[1+\frac{a_1^2}{a_{01}^2}\left(\frac{e^{-\delta_0^2/2a_{01}^2}}{\left(\frac{4\pi}{N_1\Lambda_{12}}-\frac{\Lambda_{11}}{\Lambda_{12}}\right)}-1\right)\right]\right\}^{1/2}, \quad (77)$$

which consists of a turning point beyond which the interaction between waves is attractive and conservative. The relative centroid velocity changes its sign together with $\dot{a}_1$ at the coordinates $\vec{\delta}(t)=\pm\vec{\delta}_*$, so that the beams move towards the point of coalescence ($\vec{\delta}=\vec{0}$) where they pass through each other. The beams can then periodically change place before, e.g., collapse occurs. For comparison, if $a_1(t)=a_1(0)$ was fixed, $\delta(t)$ would evolve in the range $\delta(t)\leqslant\delta_*\equiv\delta_0$ and the wave controids would oscillate periodically with a fixed period $T_p=\sqrt{2}\int_0^{\delta_*}d\delta/\sqrt{-\Pi(\delta,a_0)}$ [21]. In the present context where $a_1(t)$ varies, $\vec{\delta}(t)$ oscillates either while its amplitude and δ_* increase with $a_1(t)$ when $\delta_*\geqslant\delta_0>\delta_c$ [see Fig. 2(b)] or until vanishing when collapse occurs, as both beams merge symmetrically when $\delta_*\geqslant\delta_c\geqslant\delta_0$ [see Fig. 2(c)]. In this situation, as $a_1(t)\to 0$, the modulus of $\vec{\delta}_*$ simplifies into the limit

$$\delta_*(t)\underset{a_1(t)\to 0}{\longrightarrow}\sqrt{2}a_1(t)\ln^{1/2}\frac{1}{\left(\frac{4\pi}{N_1\Lambda_{12}}-\frac{\Lambda_{11}}{\Lambda_{12}}\right)}, \quad (78)$$

which makes sense in the regime of intermediate powers for coalescence: $4\pi/(\Lambda_{11}+\Lambda_{12})<N_1<4\pi/\Lambda_{11}$ ($0<4\pi/N_1\Lambda_{12}-\Lambda_{11}/\Lambda_{12}<1$). In this range the oscillation amplitudes are forced to decrease with $\delta(t)\leqslant\delta_*(t)\simeq O(a_1(t))$ and to attain ultimately zero, as observed in Fig. 2(c). Before this, both $\delta(t)$ and $a_1(t)$ reach extremal values at some points $\delta=\delta_*$. The number of oscillations increases with the collapse time, as the value of $\delta(0)$ is augmented to $\delta_c\simeq 2.61\rho$. Conversely, we could reason in terms of $a \leq a_*$ above which no solution exists and below which $a(t)$ oscillates until reaching zero. From a mechanical analogy, the potential function Π here plays the role of a potential well with a typical width decreasing to zero in case of collapse, in which a particle periodically moves down the potential slope towards the point of coalescence $\delta = 0$. Such oscillations might, however, be subdued in the true coalescence process because the variational method captures the total power engaged in both waves without permitting them to evacuate a power excess to the boundaries, as any NLS solution usually does with a power above the collapse threshold [28]. Note finally that the existence of the turning point δ_* at high power levels $N_1 > 4\pi/\Lambda_{11}$ does not make sense, which justifies why no oscillation develops in Fig. 2(d).

Let us now discuss some estimates of the finite collapse moment at which two coupled waves may blow up in the 2D case. When the virial integral (32) tends to vanish in finite time, both of the positive virial contributions $I_\alpha(t) \equiv N^{-1}\int(\vec{x} - \langle\vec{x}\rangle)^2|u_\alpha|^2 d\vec{x}$ tend to zero in the same limit. All the waves should thus blow up at the same instant t_c, which appears logical by virtue of their mutual nonlinear coupling in Eq. (26). Claiming this just consists of a conjecture since we know that for the cubic NLS equation a wave blow-up generally arises earlier than a total collapse for which the moment of the singularity is given by the first zero of $I(t)$. This moment $t_c^{\max}$ is the maximum time for blow-up (48), established from the virial relation $\partial_t^2 I(t) = 8H/N$ with $N = \Sigma_\alpha N_\alpha$ and H given by Eq. (39). Considering two waves without any initial divergence $[\dot{I}(0) = 0]$ and all centered on the origin, one has $I(0) = \rho^2$ and the maximum collapse time reads

$$t_c^{\max} = \rho^2 \left[\frac{N}{4\sum_\alpha N_\alpha \left\{ \sum_\beta \frac{\Lambda_{\alpha\beta} N_\beta}{4\pi} - 1 \right\}} \right]^{1/2}. \tag{79}$$

For $\Lambda_{11} = \Lambda_{22} = \frac{1}{8}$ and $\Lambda_{12} = \Lambda_{21} = \frac{1}{4}$, this expression simplifies into $t_c^{\max} = (\rho^2/2)[32\pi/(N - 32\pi)]^{1/2}$ in the one-wave case with $N = N_1 > N_c^f = 32\pi$ $(N_2 = 0)$ and into $t_c^{\max} = (\rho^2/2)[32\pi/(3N_\alpha - 32\pi)]^{1/2}$ for two superimposed waves with $N_\alpha = N_1 = N_2 > N_c^0 = 32\pi/3$. These values of $t_c^{\max}$ are in perfect agreement with those given by numerical integrations of the variational equations (71) and (70), respectively. This excellent agreement follows from the fact that for NLS systems, the behavior of the radius $a_\alpha(t) \sim \sqrt{I_\alpha(t)}$ is self-consistent with the one obtained from the virial integral [28]. For two identical but initially separated waves with $N_1 = N_2$ $(N = 2N_1)$, $\rho = a_1(0) = a_2(0) = 1$, and $\delta(0) = \delta_{\alpha\beta}(0)$, the estimate (48) reads

$$t_c^{\max} = \frac{\sqrt{I(0)}}{2} \left[\frac{N_1}{32\pi} (1 + 2e^{-\delta^2(0)/2}) - 1 \right]^{-1/2}, \tag{80}$$

where $I(0) \equiv \rho^2 + \Sigma_\alpha N_\alpha \langle\vec{x}_\alpha(0)\rangle^2/N$ reduces to $I(0) = \rho^2 + \delta^2(0)/4$ for wave forms symmetrically located from the origin. Comparing this estimate with the results obtained from the variational approach, we find $t_c^{\max} = 3.2105$ for high-power waves collapsing individually with $N_1 = N_2 = 35\pi$ and $\delta(0) = 3.5$, while numerically the variational model yields the collapse time $t_c^{\text{var}} = 1.7644$. For the same power values, but choosing now $\delta(0) = 3$, one has $t_c^{\max} = 2.6235 = t_c^{\text{var}}$. In the amalgamation regimes starting from powers $N_1 < N_c^f = 32\pi$, one can point out the characteristic values $N_1 = 30\pi$ and $\delta(0) = 2.5$ leading to $t_c^{\max} = 5.6764 \simeq t_c^{\text{var}} = 5.6756$ and $N_1 = 30\pi$ and $\delta(0) = 2.2$ yielding $t_c^{\max} = 2.3023 = t_c^{\text{var}}$. In these configurations, the collapse moment t_c^{var} is generally equal to its virial counterpart $t_c^{\max}$. This comparison holds except in the first configuration, for which the total mean square radius (from which the definition of $t_c^{\max}$ follows) and the separation distance $\delta(t)$ do not vanish simultaneously when both wave packets separately collapse. The instant $t_c^{\max}$ must here be regarded as a maximum existence time for solutions that blow up on their respective centroids before $I(t)$ has time to vanish at $t = t_c^{\max}$. So, in this situation, the results given by the variational approach have to be considered with caution.

In summary, the variational method provides reliable results fitting the virial estimates in most of configurations, the simplest of which are the one-wave case and the case of two superimposed waves with a wave radius $\sqrt{I(t)} \sim a_1(t) = a_1(0)[1 + 4\Omega t^2/a_1^4(0)]^{1/2}$, where $\Omega \equiv 1 - N_1/32\pi$ in the former situation and $\Omega \equiv 1 - 3N_1/32\pi$ in the latter one. In more complicated configurations involving some dynamics in the separation distance $\delta_{\alpha\beta}(t)$, the results obtained from the variational model may supply less precise information about the coalescence mechanism. In particular, the development of oscillations in $\delta_{\alpha\beta}(t)$, which means that before reaching a single-wave-form state waves pass through each other periodically, should be confirmed by direct numerical integrations of Eq. (26). In spite of this reserve, we can nevertheless emphasize that the variational approach restores the principal tendencies that two coupled NLS waves can develop: their simple spreading, their mutual coalescence until collapse, or their individual self-focusing according to their individual powers, as illustrated by Fig. 2.

C. Interaction potential for the self-attraction of sech-shaped waves

The previous analysis can be repeated with test functions different from Gaussians, as, e.g., sech functions. Expressed in radially symmetric geometry, sech functions yield a critical power for collapse closer to the minimal bound $N_c = 11.68$, which the L^2 norm of solutions to the cubic NLS equation with $\Lambda_{\alpha\alpha} = 1$ must exceed to promote the collapse. However, when making use of sech functions expressed as $\text{sech}[\sqrt{(x - \langle x_\alpha\rangle)^2 + (y - \langle y_\alpha\rangle)^2}]$ in the present scope, we cannot conveniently determine the integral coefficients occurring in the average Lagrangian (53). Instead, we propose a test function based on a sech shape with separable variables and defined for the αth wave as

$$\phi_\alpha(\vec{\xi}_\alpha) = \sqrt{K_\alpha}\, \text{sech}\left(\frac{x - \langle x_\alpha(t)\rangle}{a_\alpha(t)} \right) \text{sech}\left(\frac{y - \langle y_\alpha(t)\rangle}{a_\alpha(t)} \right), \tag{81}$$

where the wave radius is assumed to be the same along the x and y directions. Also, for the sake of simplicity, we will restrict the following analysis to the coupling of two waves having identical radii $a_1(t)=a_2(t)$ and symmetrically located from the origin on the x axis with $\langle x_1(t)\rangle = -\langle x_2(t)\rangle$, while $\langle y_1(t)\rangle=\langle y_2(t)\rangle=0$. The integrals entering L then read $V_\alpha=2K_\alpha\pi^2/3$, $M_\alpha=4K_\alpha=N_\alpha$, and $D_\alpha=8K_\alpha/3$, and the self-interaction potential between waves is determined by the integral $W_{\alpha\beta}(\vec{\delta}_{\alpha\beta},a_\alpha,a_\beta)$, computed as

$$W_{\alpha\beta}(\delta_{\alpha\beta}=2\langle x_\alpha\rangle,a_\alpha=a_\beta)=\frac{32}{6}K_\alpha K_\beta F\left(\frac{2\langle x_\alpha(t)\rangle}{a_\alpha(t)}\right) \tag{82}$$

for two symmetric waves with equal radii identified by the indices $\alpha=1$ and $\beta=2$. Here the function

$$F(X)=\frac{X\cosh(X)-\sinh(X)}{\sinh^3(X)} \tag{83}$$

satisfies $\lim_{X\to\pm\infty}F(X)=0$ and $\lim_{X\to 0}F(X)=\frac{1}{3}$. From the expression of the free part of the Hamiltonian (59), we deduce that 2D identical sech-shaped pulses have the critical power threshold for collapse: $N^f_{c,\mathrm{sech}}=12/\Lambda_{\alpha\alpha}$. This value is thus closer to the minimum L^2 norm $N_c/\Lambda_{\alpha\alpha}=11.68/\Lambda_{\alpha\alpha}$ for collapse than its Gaussian counterpart $4\pi/\Lambda_{\alpha\alpha}$. Moreover, when two sech-shaped pulses initially overlap with no chirp and a zero velocity in their centroid, we observe from the total Hamiltonian that each of both superimposed waves must possess a power exceeding $N^0_{c,\mathrm{sech}}=12/(\Lambda_{\alpha\alpha}+\Lambda_{\alpha\beta})$ for the two pulses to undergo a collapse. From these properties, we can guess that the regimes of interaction between separate sech pulses with $N_\alpha=N_\beta$ will be the same as those for Gaussian pulses, namely, (i) wave spreading when $N_\alpha<N^0_{c,\mathrm{sech}}$, (ii) dispersion or amalgamation of two merging pulses with medium power $N^0_{c,\mathrm{sech}}<N_\alpha<N^f_{c,\mathrm{sech}}$, whenever their initial separation distance is above or below a critical value, respectively, and (iii) individual collapses when both pulses are well separated with $N_\alpha>N^f_{c,\mathrm{sech}}$. These typical behaviors can be checked from a direct numerical integration of the variational equations for the separation distance between waves and the pulse widths, which read in this context ($N_\alpha=N_\beta$) as

$$\ddot{\delta}=\frac{4N_\alpha\Lambda_{\alpha\beta}}{3a_\alpha^4}\frac{d}{dX}F(X)\Big|_{X=\delta/a_\alpha},\quad \delta(t)\equiv 2\langle x_\alpha(t)\rangle, \tag{84}$$

$$\frac{\pi^2}{16}\ddot{a}_\alpha=\left\{1-\frac{\Lambda_{\alpha\alpha}N_\alpha}{12}\right\}\frac{1}{a_\alpha^3}-\frac{\Lambda_{\alpha\beta}N_\alpha}{4a_\alpha^3}\frac{d}{dX}[XF(X)]\Big|_{X=\delta/a_\alpha}. \tag{85}$$

In particular, periodic motions of the wave central positions and widths can be detected in the amalgamation regimes for which both $\delta(t)$ and $a_\alpha(t)$ tend to zero in finite time. Only the shape of the pulses and their related critical power threshold for collapse slightly change, compared to Gaussian test functions. The evolution regimes of coupled waves remain comparable for Gaussians and sech pulses. For further comparison, we have plotted in Fig. 3 the integral $W_{\alpha\beta}(X)$ characterizing the self-attraction potential between two waves with equal radius $a_\alpha=a_\beta$, when they are either Gaussian (solid line) or sech shaped (dashed line), as functions of the ratio $X=\delta(t)/a_\alpha(t)$. This integral provides a measure of the attractor of mutually coupled waves. It is given by $W^{\mathrm{Gauss}}_{\alpha\beta}(X)=(N_\alpha N_\beta/2\pi)\exp(-X^2/2)$ for Gaussian pulses and by $W^{\mathrm{sech}}_{\alpha\beta}(X)=(N_\alpha N_\beta/3)F(X)$ with $F(X)$ defined by Eq. (83) for sech-shaped pulses. As can be seen from Fig. 3, both attractor potentials present similar behaviors along the ratio $\delta(t)/a_\alpha(t)$: $W_{\alpha\beta}(X)$ vanishes as $X\to+\infty$, that is, when $\delta(t)\to+\infty$ or $a_\alpha(t)\to 0$ if both waves remain well separated and $W_{\alpha\beta}(X)$ reaches its maximum value as $X\to 0$, i.e., when waves mutually overlap. In this situation, the interaction term of the Hamiltonian H^{int}, which involves this integral contribution $W_{\alpha\beta}$, diverges to $-\infty$ when collapse develops with $a_\alpha(t)=a_\beta(t)\to 0$. Finally, we can note that self-attraction of waves is efficient within a broader, although less deep, basin of attraction for sech pulses than for Gaussian pulses.

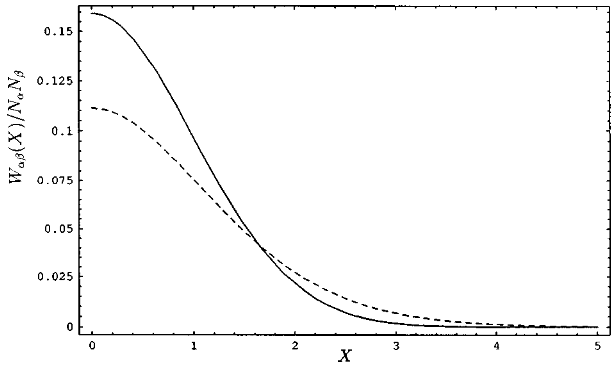

FIG. 3. Integral $W_{\alpha\beta}(X)$ versus the ratio $X=\delta(t)/a_\alpha(t)$ entering the interaction potential of two waves with equal radius $a_\alpha(t)=a_\beta(t)$, when modeling them with Gaussians (solid line) and sech functions (dashed line).

IV. AN INSTABILITY CRITERION FOR COUPLED SOLITONS

In Ref. [23] the modulational instability of two or more coupled plane waves was investigated by means of a perturbation analysis applied to Eq. (3). This analysis consisted in determining the growth rate of perturbations acting on elementary solutions to Eq. (3) that are assumed to be uniform in space and oscillatory in time as

$$A_\alpha(\vec{x},t)=A^0_\alpha\exp\left[i\sum_{\beta=1}^{n}\Lambda_{\alpha\beta}|A_\beta|^2t\right],\quad \alpha=1,...,n, \tag{86}$$

with real background components A^0_α. Instead of plane waves with uniform amplitudes, we can attempt to determine the conditions for instability of stationary solutions to Eq. (26), being bounded in space and localized with $|u_\alpha(\vec{x},t)|\to 0$ as $|\vec{x}|\to+\infty$. Such solutions usually refer to solitary waves or ''solitons'' and they are expressed as

$$u_\alpha(\vec{x},t)=R_\alpha(\vec{x})\exp(i\lambda_\alpha t). \tag{87}$$

Here R_α is supposed to be a real, positive, and even (bell-shaped) function obeying the differential equation

$$-\lambda_\alpha R_\alpha + \vec{\nabla}^2 R_\alpha + \sum_{\beta=1}^{n} \Lambda_{\alpha\beta} R_\beta^2 R_\alpha = 0 \tag{88}$$

provided $\lambda_\alpha > 0$ whenever the functions R_β ($\beta = 1,...,n$) decay to zero at infinity. For the sake of simplicity, we suppose that for each wave there exists such a stationary state, which is unique, radially symmetric, and positive at a given frequency λ_α. So we now turn to the problem of the stability of spherically symmetric solutions of Eq. (26) in the form (87). This stability problem is investigated from the perturbed solutions

$$u_\alpha = [R_\alpha + (v_\alpha + i w_\alpha)] e^{i\lambda_\alpha t}, \quad R_\alpha \equiv R_\alpha(|\vec{x}|) \tag{89}$$

where v_α and w_α are real-valued functions localized at infinity. Linearizing Eq. (26) with respect to these functions then yields the eigenvalue problem for v_α and w_α,

$$\partial_t v_\alpha = L_{0\alpha} w_\alpha, \tag{90}$$

$$-\partial_t w_\alpha = L_{1\alpha} v_\alpha - 2 \sum_{\beta \neq \alpha} \Lambda_{\alpha\beta} R_\alpha R_\beta v_\beta, \tag{91}$$

where the self-adjoint operators $L_{0\alpha}$ and $L_{1\alpha}$ are defined by

$$L_{0\alpha} \equiv \lambda_\alpha - \vec{\nabla}^2 - \sum_{\beta=1}^{n} \Lambda_{\alpha\beta} R_\beta^2, \tag{92}$$

$$L_{1\alpha} \equiv L_{0\alpha} - 2\Lambda_{\alpha\alpha} R_\alpha^2. \tag{93}$$

To describe the instability of n solitons mutually coupled with symmetric nonlinearity coefficients $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$, we define the perturbation vectors $\vec{v} = (v_1, v_2, \ldots, v_n)^T$ and $\vec{w} = (w_1, w_2, \ldots, w_n)^T$ embedding the n components of the perturbative eigenmodes (the superscript T means ''transpose'') and reformulate the above spectral problem as

$$\partial_t \vec{v} = L_0 \vec{w}, \quad -\partial_t \vec{w} = L_1 \vec{v}. \tag{94}$$

Equations (94) involve the $n \times n$ symmetric matrices, which still consist of self-adjoint operators:

$$L_0 \equiv \begin{pmatrix} L_{01} & 0 & \cdots & 0 \\ 0 & L_{02} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & L_{0n} \end{pmatrix}, \quad L_1 \equiv \begin{pmatrix} L_{11} & -2\Lambda_{12} R_1 R_2 & \cdots & -2\Lambda_{1n} R_1 R_n \\ -2\Lambda_{21} R_2 R_1 & L_{12} & \cdots & -2\Lambda_{2n} R_2 R_n \\ \cdots & \cdots & \cdots & \cdots \\ -2\Lambda_{n1} R_n R_1 & -2\Lambda_{n2} R_n R_2 & \cdots & L_{1n} \end{pmatrix}, \tag{95}$$

whose n-dimensional eigenvectors with eigenvalue zero are $\vec{R} = (R_1, R_2, \ldots, R_n)^T$ and $\nabla \vec{R} \equiv (\nabla R_1, \nabla R_2, \ldots, \nabla R_n)^T$ since $L_0 \vec{R} = \vec{0}$ and $L_1 \nabla \vec{R} = \vec{0}$. The latter relation simply follows from deriving the ground-state equation $L_{0\alpha} R_\alpha = 0$ for each $R_\alpha(|\vec{x}|)$ with respect to space variables [by convention we note $\nabla \equiv (\vec{x}/|\vec{x}|) \cdot \vec{\nabla}$]. Combining Eqs. (94), we obtain $\partial_t^2 \vec{v} = -L_0 L_1 \vec{v}$, so that when one assumes that the perturbation grows exponentially with a growth rate γ, we find that γ obeys the relation

$$\gamma^2 = -\frac{\langle \vec{v} | L_1 \vec{v} \rangle}{\langle \vec{v} | L_0^{-1} \vec{v} \rangle}. \tag{96}$$

Equation (96) makes sense provided the vectors $\vec{v}$ are orthogonal to the ground-state vector $\vec{R}$, i.e., $\langle \vec{v} | \vec{R} \rangle = 0$. Here the angular brackets $\langle \, | \, \rangle$ correspond to the L^2 inner (scalar) product between two real vectors: $\langle \vec{a} | \vec{b} \rangle = \int \vec{a}^T \cdot \vec{b} \, d\vec{x}$. Choosing $\vec{v} \perp \vec{R}$, we first recall that $\langle \vec{v} | L_0^{-1} \vec{v} \rangle$ is positive definite. Indeed, each component R_α is the unique eigenstate of $L_{0\alpha}$ with eigenvalue zero, which is the lowest eigenvalue as each R_α is positive and nodeless. Consequently, for all eigenvectors $\vec{v} \perp \vec{R}$, L_0 is positive definite and so is $\langle \vec{v} | L_0^{-1} \vec{v} \rangle$. This property can easily be found again by using the explicit formulation of $L_{0\alpha} \equiv -(1/R_\alpha) \nabla \cdot [R_\alpha^2 \nabla (1/R_\alpha)]$.

Next we determine the sign of $\langle \vec{v} | L_1 \vec{v} \rangle$ by maximizing γ^2 on the class of vectors orthogonal to $\vec{R}$. This amounts to solving the spectral problem from L_1, rewritten in the convenient form

$$L_1 \vec{v} = \lambda^* \vec{v} + \mu \vec{R}, \tag{97}$$

where the sign of λ^* will indicate stability ($\lambda^* > 0$) or instability ($\lambda^* < 0$) and $\mu \neq 0$ is an undetermined Lagrange multiplier related to the orthogonality constraint $\langle \vec{v} | \vec{R} \rangle = 0$. Following the procedure expounded in [4,33,34], we expand $\vec{v}$ and $\vec{R}$ in terms of a complete orthonormalized system of eigenvectors for the operator L_1 (which is allowed since L_1 is self-adjoint)

$$L_1 | \vec{\psi}_k \rangle = \overline{\lambda}_k | \vec{\psi}_k \rangle, \tag{98}$$

such as $|\vec{v}\rangle = \Sigma_k c_k |\vec{\psi}_k\rangle$. Elementary projections provide the coefficients $c_k = \mu \langle \vec{\psi}_k | \vec{R} \rangle / (\overline{\lambda}_k - \lambda^*)$ and therefrom

$$|\vec{v}\rangle = \mu \sum_k \frac{|\vec{\psi}_k\rangle \langle \vec{\psi}_k | \vec{R} \rangle}{\overline{\lambda}_k - \lambda^*}. \tag{99}$$

Furthermore, the orthogonality condition $\langle \vec{v} | \vec{R} \rangle = 0$ gives $\mu f(\lambda^*) = 0$ with

$$f(\lambda^*) \equiv \sum_k \frac{\langle \vec{R}|\vec{\psi}_k\rangle\langle\vec{\psi}_k|\vec{R}\rangle}{\overline{\lambda}_k - \lambda^*}. \quad (100)$$

In this sum, the eigenvector $\vec{\psi}_1 = \nabla \vec{R}$ with eigenvalue $\overline{\lambda}_1 = 0$ does not contribute since $\langle\vec{\psi}_1|\vec{R}\rangle = 0$, which leads to $c_1 = 0$. Noticing that the even components of $\vec{R}$ without node imply that the components of $\nabla\vec{R}$ each have one node, we infer that L_1 has *at least* one eigenvalue that is strictly negative $\overline{\lambda}_0 < \overline{\lambda}_1 = 0$ and it satisfies $L_1\vec{\psi}_0 = \overline{\lambda}_0\vec{\psi}_0$, where $\vec{\psi}_0$ has n components with no zeros. The existence of at least one negative eigenvalue for L_1 can follow from, e.g., differentiating the bound-state equation $L_{0\alpha}R_\alpha = 0$ with respect to the space radius $r \equiv |\vec{x}|$, as the Laplacian operator $\vec{\nabla}^2$ in Eq. (92) reduces to $\vec{\nabla}^2 = r^{1-D}\partial_r r^{D-1}\partial_r$ for radially symmetric bound states. Doing this, one readily gets $L_1\partial_r\vec{R} = [(1-D)/r^2]\partial_r\vec{R}$, with $\partial_r\vec{R} \equiv (\partial_r R_1, \partial_r R_2, \ldots, \partial_r R_n)^T$, from which we infer that L_1 has surely a negative eigenvalue when $D > 1$ since

$$\langle \partial_r\vec{R}|L_1\partial_r\vec{R}\rangle = (1-D)\left\langle \frac{1}{r}\,\partial_r\vec{R}\right|\left.\frac{1}{r}\,\partial_r\vec{R}\right\rangle. \quad (101)$$

An important remark is that, unlike the cubic NLS equation for one wave, it is not proved actually that each perturbative component in the matricial spectral problem (98) possesses a negative eigenvalue. Indeed, unlike the one-wave case for which L_1 is a scalar operator with a single negative eigenvalue, each of the n spectral equations here contains coupling terms involving the neighboring R_i's, which seems not tractable analytically. In addition, the set of all eigenvalues may not reduce to a unique value with multiplicity equal to 1. Therefore, knowing the existence of at least one negative eigenvalue, we choose $\overline{\lambda}_0 < 0$ as the largest one among all the possible negative eigenvalues ($\overline{\lambda}_0 = \sup\{\overline{\lambda}_k < 0\}$). By choosing $\overline{\lambda}_0$ so, it can be seen that $f(\lambda^*)$ decreases to $-\infty$ as $\lambda^* \to \overline{\lambda}_0$ and increases to $+\infty$ in the limit $\lambda^* \to \overline{\lambda}_2$, where $\overline{\lambda}_2$ is the first positive eigenvalue of L_1 (the latter surely exists and can be selected as, e.g., the smallest positive value in the continuum spectrum of L_1). As, moreover, $f(\lambda^*)$ is a monotonically increasing function of λ^*, it goes across the λ^* axis only once within the interval $]\overline{\lambda}_0, \overline{\lambda}_2[$, so that we can determine the sign of $\lambda^* \neq 0$ from that of $f(0)$. For $f(0) > 0$, λ^* is negative and a sufficient condition for instability follows. For $f(0) < 0$, λ^* may be positive only if it can make be sure that $\overline{\lambda}_0 < 0$ is unique. From Eq. (98) we remark that

$$f(0) = \sum_k \frac{\langle\vec{R}|\vec{\psi}_k\rangle\langle\vec{\psi}_k|\vec{R}\rangle}{\overline{\lambda}_k} = \langle\vec{R}|L_1^{-1}\vec{R}\rangle \quad (102)$$

and we can derive the ground-state equation $L_{0\alpha}R_\alpha = 0$ with respect to the frequency λ_α to get

$$L_{1\alpha}\frac{\partial R_\alpha}{\partial\lambda_\alpha} - 2\sum_{\beta\neq\alpha}\Lambda_{\alpha\beta}R_\beta R_\alpha\frac{\partial R_\beta}{\partial\lambda_\alpha} = -R_\alpha. \quad (103)$$

As it can be justified for symmetric waves or by performing a simple shift in the frequency range, we henceforth assume that all solitary waves possess an identical frequency $\lambda_1 = \lambda_2 = \cdots = \lambda_n \equiv \lambda > 0$ for $\alpha = 1,\ldots,n$. Straightforward explicit calculations based on Pohoz'aev identities can indeed show that such localized solitary waves may exist provided $D < 4$. Under this requirement, Eq. (103) takes the matricial form

$$L_1\frac{\partial\vec{R}}{\partial\lambda} = -\vec{R} \quad (104)$$

and from Eqs. (104) and (102) we finally obtain

$$f(0) = \langle\vec{R}|L_1^{-1}\vec{R}\rangle = -\left\langle\vec{R}\left|\frac{\partial\vec{R}}{\partial\lambda}\right.\right\rangle = -\frac{1}{2}\sum_{\alpha=1}^{n}\frac{\partial}{\partial\lambda}\langle R_\alpha^2\rangle,$$

leading to

$$f(0) = -\frac{1}{2}\frac{\partial}{\partial\lambda}N\{\vec{R}\}, \quad N\{\vec{R}\} \equiv \sum_{\alpha=1}^{n}\|R_\alpha\|_2^2. \quad (105)$$

Now we make use of the dilation invariance $R_\alpha(\vec{x},\lambda) \to \sqrt{\lambda}R_\alpha^0(\sqrt{\lambda}\vec{x})$ to find $N\{R_\alpha\} = \lambda^{1-D/2}N\{R_\alpha^0\}$, where $R_\alpha^0 = R_\alpha(\lambda = 1)$. From Eq. (105) it is clear that instability of bound states arises in the dimensional cases $D > 2$, for which $(\partial/\partial\lambda)N\{\vec{R}\} < 0$ assures $f(0) > 0$. Following this procedure, the dimensional case $D = 2$ consists of a marginal configuration suggesting instability.

In the opposite case $f(0) < 0$ concerning low spatial dimension numbers $D < 2$, we recall that λ^* is positive whenever L_1 has a unique negative eigenvalue solving Eq. (98). This property then supplies a necessary condition for the stability of ground states. It can also be viewed as a sufficient condition for soliton stability in the Lyapunov sense, following which $\lambda^* > 0$ ensures the positiveness of the NLS Lyapunov functional [28] $\mathcal{S} \equiv H - H\{\vec{R}\} + \lambda(N - N\{\vec{R}\}) = \langle\vec{v}|L_1\vec{v}\rangle + \langle\vec{w}|L_0\vec{w}\rangle$, with H given by Eq. (27). However, as warned above, stability follows from uniqueness and simplicity of the negative eigenvalue $\overline{\lambda}_0$, which we cannot prove due to the vectorial nature of the spectral problem (94). Let us indeed imagine that a second discrete eigenvalue $\overline{\lambda}_0' < \overline{\lambda}_0 < 0$ exists; then, within the interval $]\overline{\lambda}_0', \overline{\lambda}_0[$, there should exist another value λ^* located before the root of $f(0)$, ensuring the instability of solitary waves even in the 1D case. Also, if L_1 has a double eigenvalue, there exist two orthogonal eigenstates $\vec{v}_1$ and $\vec{v}_2$ that can be linearly combined to construct a vector $\vec{v}$ perpendicular to $\vec{R}$, which surely leads to instability. This ambiguity is nevertheless overcome when one investigates the stability of identical ground states in the form $R_1(|\vec{x}|) = R_2(|\vec{x}|) = \cdots = R_n(|\vec{x}|) \equiv \phi(|\vec{x}|)$, where ϕ obeys the cubic differential equation

$$-\lambda\phi + \vec{\nabla}^2\phi + (\Lambda_{11} + \Lambda_{12} + \cdots + \Lambda_{1n})\phi^3 = 0, \quad (106)$$

by applying the symmetry between the coupling coefficients $\Sigma_{\alpha,\beta}\Lambda_{\alpha\beta} = \Sigma_{\beta,\alpha}\Lambda_{\beta\alpha} > 0$ with $\alpha,\beta = 1,2,\ldots,n$. In that case, L_1 is a scalar operator with only one negative eigenvalue $\overline{\lambda}_0$

<0, so that the constraint $f(0)<0$ leading to $\partial N/\partial\lambda>0$ guarantees the stability of the coupled ground states. This property naturally applies to 1D NLS systems for which coupled solitons compose the set of minimizers of Eq. (26). This set decomposes over the elementary sech soliton solution

$$\phi(x)=\sqrt{\frac{2\lambda}{\sum_{\alpha=1}^{n}\Lambda_{1\alpha}}}\,\mathrm{sech}(\sqrt{\lambda}x), \qquad (107)$$

which is thus stable. This result agrees with the orbital stability of two coupled NLS ground states demonstrated in Ref. [35] for $D=1$.

Finally, we can here emphasize that the growth rate (5) for plane-wave instability can be refound in the case of, e.g., two coupled waves, when combining Eqs. (90) and (91) for uniform states $R_\alpha=A_\alpha$ ($\nabla^2R_\alpha=0$) with frequencies $\lambda_\alpha=\Sigma_{\beta=1,2}\Lambda_{\alpha\beta}|A_\beta|^2$, which become unstable under oscillatory perturbations evolving as $v_\alpha,w_\alpha\sim\cos(\vec{k}\cdot\vec{x})$. In addition, the preceding analysis could easily be repeated for a general nonlinearity function $\mathcal{F}(\cdot)$ entering $\Sigma_{\beta=1}^{n}\Lambda_{\alpha\beta}\mathcal{F}(|u_\beta|^2)u_\alpha$ with $\mathcal{F}(s)\to 0$ as $s\to 0$, instead of a cubic one, $\Sigma_{\beta=1}^{n}\Lambda_{\alpha\beta}|u_\beta|^2u_\alpha$, in Eq. (26). In that case, the basic operator $L_{0\alpha}$ originally given by Eq. (92) for cubic nonlinearities has to be replaced by $L_{0\alpha}^{\mathcal{F}}=\lambda_\alpha-\vec{\nabla}^2-\Sigma_{\beta=1}^{n}\Lambda_{\alpha\beta}\mathcal{F}(R_\beta^2)$, so that the functions (v_α,w_α) obey the eigenvalue problem

$$\partial_t v_\alpha=L_{0\alpha}^{\mathcal{F}}w_\alpha,$$

$$-\partial_t w_\alpha=L_{0\alpha}^{\mathcal{F}}v_\alpha-2\sum_{\beta=1}^{n}\Lambda_{\alpha\beta}\mathcal{F}'(R_\beta^2)R_\alpha R_\beta v_\beta, \qquad (108)$$

with $\mathcal{F}'(R_\beta^2)\equiv\partial\mathcal{F}/\partial R_\beta^2$.

V. DISCUSSION

In the present analysis we have investigated the various regimes of mutual interaction between nonlinear light waves described by several coupled NLS equations. Three typical regimes of interaction between two identical Gaussian waves naturally arise from this analysis. (i) When both pulses have a power below the threshold N_c^0 for the self-focusing of superimposed wave packets they spread out asymptotically in time. (ii) When the optical pulses possess an individual power between N_c^0 and N_c^f, where N_c^f is the self-focusing threshold power for an isolated Gaussian, the waves fuse into one entity that spreads out asymptotically or collapses in finite time whenever their initial separation distance is above or below a critical value δ_c, respectively. This critical distance of separation depends on the power contained in each wave and the basin of attraction delimited by δ_c increases with the wave power. (iii) When the two waves have a power exceeding N_c^f they individually collapse without mutual correlation, except when their separation distance is less than δ_c, in which case they can merge into one self-focusing structure. Here δ_c decreases with the wave power. All these characteristic regimes have been described and confirmed in Sec. III through a variational approach that restores the main dynamics inferred from the virial arguments expounded in Sec. II, up to oscillations in the separation distance $\delta(t)$. Criteria for coalescence have been elaborated on the vanishing in finite time of the virial integral $I(t)$ for negative-energy states, which includes the vanishing of the separation distance between wave centroids and depends on the number and individual power of beams. Consequently, those theoretical expectations may be altered to a certain extent by blow-up phenomena occurring before the total mean square radius of coupled waves, together with their mutual separation distance, has completely vanished. Further details on interaction dynamics require numerical integrations of Eq. (26) that will be presented elsewhere [36] and will clearly display evidence of amalgamation mechanism and development of oscillations in the wave centroids. Eventually, a sufficient condition for the instability of coupled NLS solitary waves and a stability criterion for identical coupled NLS ground states have also been constructed, which can be expressed in terms of the derivative of the soliton powers with respect to their frequency. This instability criterion is compatible with the condition for blow-up yielded by the virial identity, following which a wave collapse can develop in cubic media for $D\geqslant 2$ only.

Finally, it is worth investigating the competition between modulational instability and coalescence of coupled waves in order to know whether a mutual amalgamation of waves can be realized before the full development of their modulational instability. This property might, for example, be used to promote efficiently the formation of hot spots with a very high peak power in nonlinear, weakly dispersive media, starting with a set of incident beams with much weaker intensities.

For one wave ($A_1\neq 0$, $A_2=0$) with Λ_{11} normalized to unity, the typical time for forming filaments from a uniform plane wave with constant amplitude $|A_1|$ and initial length $L_\perp$ is given by the inverse of the growth rate derived from Eqs. (5)–(7) and reducing in that case to $\gamma_{\max}=|A_1|^2$. The number of filaments formed at $t_{\mathrm{fil}}=\gamma_{\max}^{-1}$ is equal to $\mathcal{N}=(L_\perp/\lambda_{\mathrm{mod}})^D$, where D is the space dimension number and $\lambda_{\mathrm{mod}}=2\pi/|A_1|=2\pi/k_{\max}$ [$k_{\max}=|A_1|$ maximizes the growth rate $\gamma(k)$]. In terms of power, we also have $\mathcal{N}=P_0/P_{\mathrm{fil}}$, where $P_0=P(L_\perp/2)$ and $P_{\mathrm{fil}}=P(\lambda_{\mathrm{mod}}/2)$ are computed from the integral function $P(x)\equiv 2^{D-1}\pi\int_0^x|A_1|^2r^{D-1}dr$. We can notice that the initial beam power $P_0=(\pi/4)L_\perp^2|A_1|^2=N_1$ recovers the same value when it is computed with a one-component 2D Gaussian profile selected in Eq. (37), modulo the substitution $N_1\leftrightarrow\pi\rho^2|A_1|^2$ with $\rho=L_\perp/2$. Besides, the maximal collapse time of one self-focusing beam is given by $t_c^{\max}=\sqrt{-I(0)N_1/4H}=\rho^2/2\sqrt{|A_1|^2\rho^2/4-1}$ for beams initially at rest with no initial divergence. Provided $\rho^2|A_1|^2>4$, an intense beam has totally self-focused at $t=t_c^{\max}$ and cannot exist any longer afterwards. Comparing thus $t_c^{\max}$ with t_{fil}, it is easily seen that $t_{\mathrm{fil}}<t_c^{\max}$, which indicates that the beam first produces filaments before self-focusing on the whole. Keeping this result in mind, we can then wonder whether two Gaussian beams can coalesce and collapse in the form of one self-focusing lobe before producing filaments. The key idea here consists in comparing $\gamma_{\max}^{-1}$ for two equal wave packets [Eqs. (5) and (7)] with $t_c^{\max}(\delta)$ [Eq. (48)] in the range of powers $[N_{\mathrm{crit}}/(\Lambda_{11}+\Lambda_{12})]<N_1=N_2<(N_{\mathrm{crit}}/\Lambda_{11})$ ($N_{\mathrm{crit}}=4\pi$), where waves amalgamate under the condition $\delta \leq \delta_c$ only. In the context of two identical beams ($\Lambda_{11} = \Lambda_{22}$), the time for generating $\mathcal{N} = P_0/P_{\mathrm{fil}}$ filaments from two coupled Gaussian waves is estimated by

$$t_{\mathrm{fil}} = \frac{1}{(\Lambda_{11} + \Lambda_{12})|A_1|^2} = \frac{\pi\rho^2}{(\Lambda_{11} + \Lambda_{12})N_1}, \tag{109}$$

with $N_1 = N_2 = \pi\rho^2|A_1|^2$, whereas their maximum collapse time reads

$$t_c^{\max}(\delta) = \frac{\rho^2}{2}\left\{\frac{1 + \delta^2(0)/4\rho^2}{(\Lambda_{11} + \Lambda_{12}e^{-\delta^2(0)/2\rho^2})\,\frac{|A_1|^2\rho^2}{4} - 1}\right\}^{1/2}. \tag{110}$$

Assuring $\Lambda_{12} = 2\Lambda_{11}$, one observes that $t_c^{\max}(\delta)$, although diminishing as $\delta(0) \to 0$, always remains larger than t_{fil} in the range of powers promoting coalescence

$$\frac{1}{3} < \frac{N_1\Lambda_{11}}{4\pi} < 1, \tag{111}$$

in such a way that the stage of filamentation breaking periodically the two waves might not be overcome by their mutual amalgamation.

To conclude this investigation, we would also like to emphasize some analogies between copropagating and counterpropagating waves. To this aim, let us focus our attention on a system of two counterpropagating scalar wave envelopes, which are described as

$$i\partial_t u_1 + \vec{\nabla}^2 u_1 + \Lambda_{11}|u_1|^2 u_1 + \Lambda_{21}|u_2|^2 u_1 = 0, \tag{112}$$

$$-i\partial_t u_2 + \vec{\nabla}^2 u_2 + \Lambda_{12}|u_1|^2 u_2 + \Lambda_{22}|u_2|^2 u_2 = 0, \tag{113}$$

in which u_2 evolves along the time variable in the direction opposite to u_1's. First of all, it is easy to check that both of the individual powers N_α ($\alpha = 1,2$) and the Hamiltonian integral (27) are still conserved for localized solutions of Eqs. (112) and (113). Assuming again $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$ ($\alpha,\beta = 1,2$), we can also derive the evolution equation for the partial centers of mass, reading

$$\partial_t^2\langle \vec{x}_\alpha(t)\rangle = \frac{2}{N_\alpha}\,\Lambda_{\alpha\beta}\int |u_\alpha|^2\vec{\nabla}|u_\beta|^2 d\vec{x} \quad (\alpha \neq \beta), \tag{114}$$

which is nothing but the relation (29) previously derived in the context of copropagating waves. This again yields the conservation laws about the total center of mass located in the transverse space $\partial_t\langle\vec{x}\rangle = \vec{P}/N$ and $\partial_t^2\langle\vec{x}\rangle = \vec{0}$ with $N\langle\vec{x}\rangle = N_1\langle\vec{x}_1\rangle + N_2\langle\vec{x}_2\rangle$. By repeating the principal steps for deriving the virial identity, it can be verified that the integral $I(t) = \langle(\vec{x} - \langle\vec{x}\rangle)^2\rangle$ for counterpropagating waves is governed by a dynamical relation that is identical to Eq. (33) previously established for copropagating waves. This result signifies in particular that the merging dynamics of two counterpropagating waves in their transverse diffraction plane is analogous to that taking place for two copropagating pulses. A direct consequence of this result is that the formation of a central high-intensity hot spot promoted by the coalescence of several copropagating waves could in principle be reinforced by the transfer of additional power originating from the same number of nonlinear beams propagating in the opposite direction.

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